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On Symmetric Functions and Seminvariants.

BY PROF. CAYLEY.

The principal object of the present memoir is to further develop the theory of seminvariants, but in connection therewith I was led to some investigations on symmetric functions, and I have consequently included this subject in the title. The two theories, if we adopt the MacMahon form of equation,

$$0 = 1 + bx + \frac{c}{2}x^2 + \frac{d}{6}x^3 + \ldots\ldots,$$

may be regarded as identical; but there are still two branches of the theory, viz. we may seek to obtain for the symmetric functions of the roots expressions in terms of the coefficients (which expressions, in the case of nonunitary symmetric functions, are in fact seminvariants), or we may attend to the properties of the functions of the coefficients thus obtained and which we call seminvariants. But I do not in the first instance use the MacMahon form, but retain the ordinary form of equation $0 = 1 + bx + cx^2 + dx^3 + \ldots\ldots$, and we have thus only a parallelism of the two theories, and in place of seminvariants we have functions which I call nonunitariants. In regard as well to these as to unitariant functions, I consider certain operators Θ_σ, Δ, $P - \delta b$, and $Q - 2\omega b$, which under altered forms present themselves also in the theory of seminvariants.

As regards seminvariants, I consider what I call the blunt and sharp forms respectively: the great problem is, it appears to me, that of sharp seminvariants, otherwise the I-and-F problem—viz. for any given weight we have to determine the correspondence between the initial and final terms in such wise as to obtain a system of sharp seminvariants. I obtain a "square diagram" solution, which is so far theoretically complete that for any given weight I can, without any

tentative operation, determine by a laborious process the correspondence in question: but I am not thereby enabled to establish or enunciate for successive weights any general rule of correspondence; and my process is in fact as regards practicability far inferior to that which I call the MacMahon linkage, but of the validity of this I have not succeeded in obtaining any satisfactory proof.

I establish an umbral theory of seminvariants which will be presently again referred to, and I consider the question of the reduction of seminvariants. The final term of a seminvariant may be composite (that is, the product of two or more final terms), and that in one way only or in two or more ways, or it may be non-composite. In the case of a composite final term the seminvariant is reducible, but the converse theorem that a seminvariant with a non-composite final term is irreducible is in nowise true; the reason of this is explained. An irreducible seminvariant is a perpetuant. In regard to perpetuants I reproduce and simplify a demonstration recently obtained by Dr. Stroh as to the perpetuants for any given degree whatever: viz. the generating function for perpetuants of degree n is $= x^{2^{n-1}-1} \div 1 - x^2 . 1 - x^3 \ldots . 1 - x^n$; the theorem was previously known, and more or less completely proved, for the values $n = 4, 5, 6$, and 7. Dr. Stroh's investigation is conducted by an umbral representation,

$$(\alpha x + \beta y + \gamma z + \ldots .)^n,\ x + y + z + \ldots . = 0,$$

of the blunt seminvariants of a given weight.

I consider in regard to seminvariants the theory of the symbols $P - \delta b$ and $Q - 2\omega b$, and the derived symbols Y and Z, each of which operating on a seminvariant gives a seminvariant. These are in fact connected with the derivatives (f, F) of a quantic f and any covariant thereof F, but except to point out this connexion I do not in the present memoir consider the theory of covariants.

The Coefficients $(a, b, c, d, e, \ldots .)$ *or* $(1, b, c, d, e, \ldots .)$. *Article Nos.* 1 *to* 9.

1. I consider the series $(a, b, c, d, e, \ldots .)$, or putting as we most frequently do $a = 1$, say the series $(1, b, c, d, e, \ldots .)$ of coefficients, the several terms whereof are taken to be of the weights $0, 1, 2, 3, 4, \ldots .$ respectively. We form with these sets of isobaric terms, or say columns of the weights $0, 1, 2, 3, 4, \ldots .$ respectively, for instance,

0	1	2	3	4	5	6
1	b	c	d	e	f	g
		b^2	bc	bd	be	bf
			b^3	c^2	cd	ce
				b^2c	b^2d	d^2
				b^4	bc^2	b^2e
					b^3c	bcd
					b^5	c^3
						b^3d
						b^2c^2
						b^4c
						b^6

and generally a set or column of any given weight. In each term the letters are written in alphabetical order.

Taking the whole or any part of a column, for instance the whole column (d, bc, b^3), or the part (e, bd, c^2) of the next column, we may by supplying powers of a in such wise as to leave unaltered the terms of the highest degree, that is by reading these as (a^2d, abc, b^3) and (ae, bd, c^2) respectively, regard them as homogeneous sets of a given degree in $(a, b, c, d, e, \ldots)$; and thus generally we may speak of the degree of a set of terms.

The terms of the several columns as above written down are in alphabetical order, AO; viz. we supply as above the proper powers of a, reading for instance col. 4 as a^3e, a^2bd, a^2c^2, ab^2c, b^4, where the terms are in alphabetical or dictionary order.

Each column is derived from the preceding one by Arbogast's rule, it being understood, for instance, that b^4, that is ab^4, gives the two terms ab^3c and b^5, that is b^3c and b^5; and so in other cases.

2. We attend in particular to the nonunitary terms, or nonunitaries, e. g. in col. 5, f, cd, which contain no b; and to the power-ending terms or power-enders, bc^2, b^5, which end in a power. It will be observed that whenever by Arbogast's rule a term in one column gives two terms in the next column, the second of these is a power-ender; and thus in any column the excess of the number of terms above that in the preceding column is equal to the number of power-enders.

3. I consider the notion of conjugate terms: representing, for instance, the terms f be cd by dots in the form $.....$ $:...$ $::.$ and reading the number of dots in columns instead of in lines we derive the conjugate terms b^5 b^3c bc^2, and so in other cases. It is clear that the relation is a reciprocal one (thus the conjugates of b^5, b^3c, bc^2 are f, be, cd respectively). Moreover, a term may be its own conjugate; thus cd^2, arranging the dots in lines and reading them in columns, $\vdots\ \vdots\ :$ is again cd^2.

It is at once seen that nonunitaries and power-enders are conjugate to each other; hence in any column the nonunitaries and the power-enders are equal in number, and a preceding result may be stated in the more complete form: in any column the excess of the number of terms above that in the preceding column is equal to the number of nonunitaries or to the number of power-enders.

4. The terms of the several columns may be arranged in counter-order CO, thus:

0	1	2	3	4	5	6
1	b	c	d	e	f	g
		b^2	bc	bd	be	bf
			b^3	c^2	cd	ce
				b^2c	b^2d	b^2e
				b^4	bc^2	d^2
					b^3c	bcd
					b^5	b^3d
						c^3
						b^2c^2
						b^4c
						b^6

viz. we arrange here according to the highest letters. The counter-order is in fact the alphabetical order with the reversed arrangement $(\ldots . g, f, e, d, c, b, a)$ of the alphabet, but in the separate terms we retain the alphabetical order, thus writing as before bf and not fb. Observe that the difference between the two arrangements, AO and CO, first presents itself in the col. 6.

In this CO arrangement each column is derived from the next preceding one by a rule as follows: We operate on the lowest letter of each term, being a

simple letter, not a power, by changing it into the next highest letter, and we further operate upon each term by multiplying it by b, the operation or (as the case may be) two operations upon any term being performed before operating upon the next term.

5. If we compare a column in AO with the same column in CO, for instance

AO	CO	AO	CO rev.
g	g	g	b^6
bf	bf	bf	b^4c
ce	ce	ce	b^2c^2
d^2	b^2e	d^2	c^3
b^2e	d^2	b^2e	b^3d
bcd	bcd	bcd	bcd
c^3	b^3d	c^3	d^2
b^3d	c^3	b^3d	b^2e
b^2c^2	b^2c^2	b^2c^2	ce
b^4c	b^4c	b^4c	bf
b^6	b^6	b^6	g

it will be seen that the terms are conjugates of each other, the first and last, the second and last but one terms, and so on; or what is the same thing, if we reverse the order of either column, then the pairs of conjugate terms will appear each in the same line; of course, here a self-conjugate term such as bcd is put in evidence.

6. By writing $a, b, c, d, \ldots = a_0, a_1, a_2, a_3, \ldots$, or more simply $0, 1, 2, 3, \ldots$, we connect the theory with that of the partition of numbers: in particular the terms of a given weight correspond to the partitions of that weight, or number of ways in which that weight can be made up with the parts $1, 2, 3, \ldots$. It may be remarked that in a partition the parts are usually written in decreasing order, whereas (as remarked above) in a literal term the letters are written in alphabetical order. Thus we have 321 and bcd; it would be more correct to write the partition as 123.

It is frequently convenient, retaining the letters $b, c, d, \ldots$, to write for instance $q = a_\sigma$ (σ a numerical suffix), meaning thereby that q is the letter corresponding to the place σ in the series $1, 2, 3, \ldots$. If instead of the indefinite series $(1, b, c, d, \ldots)$ we consider, as is sometimes convenient, a definite series

of terms $(1, b, c, \ldots . q = a_\sigma)$, then σ is said to be the "extent" of the system. The next preceding letter p will naturally be $= a_{\sigma-1}$; and if increasing the extent by unity we introduce a new letter r, this will be $a_{\sigma+1}$, and so in other cases, the notation being for the most part used merely as a convenient way of showing the place of a letter in the series.

7. Considering the terms of a given weight, or say a column, in AO or CO, we may represent any portion of the column by means of its initial and final terms, say I and F, by the notations Iao$F$ and Ico$F$ respectively. But a much more important notation is Ica$F$; viz. this represents the series of terms of given weight which are in CO not superior to I, and in AO not inferior to F (a like notation, which however I do not employ, would be Iac$F$; viz. this would denote the series of terms which are in AO not superior to I and in CO not inferior to F). The definition of Ica$F$ has been given in the above general form, but we are in fact exclusively or chiefly concerned with the case where I is a nonunitary and F a power-ender. It is to be observed that considering the AO column as given, then to form from it the set or interval Ica$F$ we may disregard altogether the terms which are in the AO column inferior (posterior) to F, for by the definition none of these enter into Ica$F$, but it may very well be that there are in Ica$F$ terms which are in the AO column superior (anterior) to I. An instance of this first presents itself for the weight 11; viz. here a portion of the AO column is $(fg,\ b^2j,\ bci,\ bdh,\ beg,\ bf^2,\ c^2h,\ cdg, \ldots .)$: hence in Ica$F$, if the initial term be c^2h, for instance in c^2hcab^3e^2, we have terms fg, beg, bf^2 which are in AO anterior to the initial term c^2h. In order therefore to form Ica$F$ from the AO column we must first take the terms (if any) which being in CO posterior to I are in the AO column anterior to I, and then from the portion Iao$F$ of the AO column reject the terms (if any) which are in CO anterior to I. In particular, starting from the AO column, and arranging the non-unitaries thereof in CO and the power-enders in AO, for instance, weight 12, these are

m	g^2
ck	cf^2
dj	e^3
ei	b^2f^2
c^2i	bde^2
fh	c^2e^2
$\vdots$	$\vdots$

There is no difficulty in writing down the terms of the several sets or intervals

$$mcag^2,\ mcacf^2,\ mcae^3,\ \ldots\ ckcag^2,\ ckcacf^2,\ \ldots$$

Instead of ca we may, if we please, use, and in fact I generally use the conventional symbol ∞, or write $m\infty g^2$, $m\infty cf^2$, etc. In any such set the terms need not be arranged in AO; if for any purpose it is more convenient they may be arranged in CO; but of course the definition of the meaning must not be departed from. The expressed initial is the highest term in CO, and the expressed final the lowest term in AO.

8. I diminish a term by replacing successively each letter thereof by the next inferior letter; for instance, if the term is cdf, then the diminished terms are $Dcdf, = bdf,\ c^2f,\ cde$, and so $Db^2df, = bdf,\ b^2cf,\ b^2de$ (where the diminished b is a, that is 1). Conversely we may augment a term by replacing successively each letter thereof by the next superior letter; for instance, $Abdf, = b^2df,\ cdf,\ bef,\ bdg$, where the first augmentation b^2df is obtained from the a (which may be regarded as latent in the term operated upon). Operating upon the letters in order beginning with the lowest, the several diminutions may be called $D_1, D_2, D_3, \ldots$ and the several augmentations $A_0, A_1, A_2, \ldots$ (where A_0 is in fact multiplication by b). We diminish a set by diminishing successively the several terms thereof (the diminished terms being taken without repetition; that is, each such term once only). Similarly we may augment a set by augmenting successively the several terms thereof (the augmented terms being taken without repetition). It is to be noticed that the two operations are not reciprocal to each other; if we diminish a set, and then augment the diminished set, we obtain indeed all the terms of the original set, but in general we obtain also terms which are not included in the original set.

9. It requires some consideration to see that we have $D(I\infty F) = (D_1I\infty D_\theta F)$, where $D_\theta F$ is the diminution performed upon the highest letter of F. Take any term M of $D(I\infty F)$, the several diminutions $D_1M, D_2M, \ldots D_\phi M$ are arranged in descending order: D_1M the highest and $D_\phi M$ the lowest, as well in CO as in AO. If then D_1M is in CO not superior to D_1I, then all the DM's will be in CO not superior to D_1I; and similarly if $D_\phi M$ is in AO not inferior to $D_\theta F$, then all the DM's will be in AO not inferior to $D_\theta F$. And this being seen, then if we take N a term of $(D_1I\infty D_\theta F)$, and consider the successive augmentations A_0N, $A_1N, \ldots A_\phi N$ of N, then these will be in ascending order A_0N the lowest and $A_\phi N$ the highest in CO as well as in AO. It may happen that $A_\phi N$ or this and

neighboring terms are in CO higher than I, and that A_0N or this and neighboring terms are in AO lower than F, but there will always be a term or terms which is or are in CO lower than I and in AO higher than F; and thus not only every term of $D(I \overset{\bullet}{\infty} F)$ will be a term of $(D_1 I \infty D_\theta F)$, but conversely every term of $(D_1 I \infty D_\theta F)$ will be a term of $D(I \infty F)$, and we thus have the required relation $D(I \infty F) = (D_1 I \infty D_\theta F)$.

Symmetric Functions of the Roots. Article Nos. 10 *to* 31.

10. We consider a set of roots $\alpha, \beta, \gamma, \delta, \varepsilon, \ldots.$ either indefinite in number, or else definite, for instance $\alpha, \beta, \gamma, \delta$. The symmetric functions (rational and integral functions) are in the first instance denoted in the usual manner $S\alpha = \alpha + \beta + \gamma + \delta + \ldots.$, $S\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma + \ldots.$, $S\alpha^2\beta = \alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2 + \ldots.$, viz. the S refers to all the distinct combinations of like form with the combination (α, $\alpha\beta$, or $\alpha^2\beta$, as the case may be) to which it is prefixed. By omitting the S and instead of the roots considering merely their indices, these same symmetric functions would be 1, 11 (or 1^2), 21, etc., and then if instead of the numbers 1, 2, 3, etc., we introduce the symbolic capital letters $B, C, D, \ldots.$ the same symmetric functions will be represented as B, B^2, BC, etc. (21, that is 12 is written as BC, and so in other cases, the letters in alphabetical order). The letters $B, C, D, \ldots.$ are considered as being of the weights $1, 2, 3, \ldots.$ respectively, and thus the symmetric functions of a given degree in the roots are represented by the terms of that weight in the symbolic letters $B, C, D, \ldots.$ thus the symmetric functions of the degree 4 are E, BD, C^2, B^2C, B^4; of course these terms may be arranged in AO or in CO as may be most convenient for the purpose in hand. The capital letters $B, C, D, \ldots.$ are in fact umbræ, but to avoid confusion with subsequent notations I do not in general thus speak of them. A form such as $S\alpha^2$ or $S\alpha^4\beta^2, \ldots.$ in which there is no index 1 is said to be nonunitary, but a form $S\alpha$ or $S\alpha^2\beta$ in which there is an index $= 1$ or two or more indices each $= 1$ is said to be unitary: or what is the same thing, in the symbolic representation by capital letters, the form is nonunitary or unitary according as it does not or does contain the letter B.

11. In the ordinary theory of symmetric functions we connect the coefficients $(1, b, c, d, \ldots.)$ with the roots $(\alpha, \beta, \gamma, \ldots.)$ by the equation

$$1 + bx + cx^2 + dx^3 + \ldots. = 1 - \alpha x \,.\, 1 - \beta x \,.\, 1 - \gamma x \ldots.,$$

and we thus have

$$\begin{aligned}
-b &= \alpha + \beta + \gamma + \ldots\ldots, = S\alpha, = 1, = B,\\
+c &= \alpha\beta + \alpha\gamma + \beta\gamma + \ldots\ldots, = S\alpha\beta, = 1^2, = B^2,\\
-d &= \alpha\beta\gamma + \ldots\ldots \qquad , = S\alpha\beta\gamma, = 1^3, = B^3,\\
&\text{etc., etc.};
\end{aligned}$$

and it is to be remarked, that for any given number of roots there will be this same number of coefficients: we may for instance have

$$1 + bx + cx^2 + dx^3 = 1 - \alpha x \,.\, 1 - \beta x \,.\, 1 - \gamma x, \text{ that is } \begin{aligned} -b &= \alpha + \beta + \gamma,\\ +c &= \alpha\beta + \alpha\gamma + \beta\gamma,\\ -d &= \alpha\beta\gamma, \end{aligned}$$

and similarly if the number of roots be $= 4$, or any larger number.

12. The symmetric functions of a given degree, say 4, in the roots, viz.

$$\begin{array}{cccccl}
S\alpha^4, & S\alpha^3\beta, & S\alpha^2\beta^2, & S\alpha^2\beta\gamma, & S\alpha\beta\gamma\delta, & \text{or}\\
4, & 31, & 2^2, & 21^2, & 1^4, & \text{or}\\
E, & BD, & C^2, & B^2C, & B^4 &
\end{array}$$

are equal in number to the combinations of the weight 4 in the coefficients, viz.

$$e,\quad bd,\quad c^2,\quad b^2c,\quad b^4$$

and the terms of the one set are in fact linear combinations (with mere numerical multipliers) of the terms of the other set; but more than this, we have for instance

$e = \alpha\beta\gamma\delta + \ldots\ldots$, that is $e = B^4$.

$bd = (\alpha + \beta + \gamma + \delta \ldots.)(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta \ldots.)$ contains only terms $\alpha^2\beta\gamma$, and $\alpha\beta\gamma\delta$, that is bd is a linear function of B^2C and B^4.

$c^2 = (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta \ldots.)^2$ contains only terms $\alpha^2\beta^2$, $\alpha^2\beta\gamma$ and $\alpha\beta\gamma\delta$, that is c^2 is a linear function of C^2, B^2C and B^4; and so on.

13. We have in fact the Table IV(a) which I quote from my paper "A Memoir on the Symmetric Functions of the Roots of an Equation," Phil. Trans. t. 147 (1857), pp. 489–496; Coll. Math. Papers, 147,

‖	e	bd	c^2	b^2c	b^4
$S\alpha^4 = 4 = E$					$+1$
$S\alpha^3\beta = 31 = BD$				$+1$	$+4$
$S\alpha^2\beta^2 = 2^2 = C^2$			$+1$	$+2$	$+6$
$S\alpha^2\beta\gamma = 21^2 = B^2C$		$+1$	$+2$	$+5$	$+12$
$S\alpha\beta\gamma\delta = 1^4 = B^4$	$+1$	$+4$	$+6$	$+12$	$+24$

inserting on the left-hand outside margin the new symbols E, BD, etc., with their explanations: the $\|$ indicates that the table is to be read according to the columns, $e = +1B^4$, $bd = +1B^2C + 4B^4$, etc. This table gives conversely a table IV(b) read according to the lines and serving to express the symmetric functions E, BD, etc., as linear functions of the combinations e, bd, c^2, b^2c, b^4 of the coefficients.

14. The (a) and (b) tables are given in the Memoir up to X(a) and X(b): it is proper to quote here the (b) tables up to VI(b) with only the change of substituting on the outside left-hand margins the literal terms such as E, BD, etc., instead of the symbols 4, 31, etc., originally used to denote these symmetric functions—it is to be observed that the left-hand symbols are in AO, the upper symbols in CO, this distinction first manifesting itself in the table VI(b), so that it was necessary to go as far as this in order to put in evidence the true form of the tables.

II(b).

=	c	b^2
C	−2	+1
B^2	+1	

III(b).

=	d	bc	b^3
D	−3	+3	−1
BC	+3	−1	
B^3	−1		

IV(b).

=	e	bd	c^2	b^2c	b^4
E	−4	+4	+2	−4	+1
BD	+4	−1	−2	+1	
C^2	+2	−2	+1		
B^2C	−4	+1			
B^4	+1				

V(b).

=	f	be	cd	b^2d	bc^2	b^3c	b^5
F	−5	+5	+5	−5	−5	+5	−1
BE	+5	−1	−5	+1	+3	−1	
CD	+5	−5	+1	+2	−1		
B^2D	−5	+1	+2	−1			
BC^2	−5	+3	−1				
B^3C	+5	−1					
B^5	−1						

VI(b).

=	g	bf	ce	b^2e	d^2	bcd	b^3d	c^3	b^2c^2	b^4c	b^6
G	−6	+6	+6	−6	+3	−12	+6	−2	+9	−6	+1
BF	+6	−1	−6	+1	−3	+7	−1	+2	−4	+1	
CE	+6	−6	+2	+2	−3	+4	−2	−2	+1		
D^2	+3	−3	−3	+3	+3	−3	0	+1			
B^2E	−6	+1	+2	−1	+3	−3	+1				
BCD	−12	+7	+4	−3	−3	+1					
C^3	−2	+2	−2	0	+1						
B^3D	+6	−1	−2	+1							
B^2C^2	+9	−4	+1								
B^4C	−6	+1									
B^6	+1										

It is hardly necessary to remark in relation to these tables that if there are only two roots, then $d=0$, etc., viz. Table II is not affected but all the subsequent tables assume a simplified form; if there are only three roots, then $e=0$, etc., viz. Tables II and III are not affected but all the subsequent tables assume a simplified form; and so on.

15. We have between the differential symbols ∂_b, ∂_c, ∂_d, ... and ∂_α, ∂_β, ∂_γ, ... certain relations which it is interesting to develop: it will be convenient to consider successively the cases, three roots, four roots, etc.

In the case of three roots, starting from

$$\begin{aligned} -b &= \alpha + \beta + \gamma, \\ c &= \alpha\beta + \alpha\gamma + \beta\gamma, \\ -d &= \alpha\beta\gamma, \end{aligned}$$

we have

$$\begin{aligned} \partial_\alpha &= -\partial_b + (\beta+\gamma)\partial_c - \beta\gamma\partial_d, \\ \partial_\beta &= -\partial_b + (\gamma+\alpha)\partial_c - \gamma\alpha\partial_d, \\ \partial_\gamma &= -\partial_b + (\alpha+\beta)\partial_c - \alpha\beta\partial_d, \end{aligned}$$

equations which give conversely ∂_b, ∂_c, ∂_d as linear functions of ∂_α, ∂_β, ∂_γ: I write down the three equations thus obtained together with a fourth equation which I will explain. The four equations are

$$-\partial_a+\delta'=\frac{\alpha^3}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^3}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^3}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_b\quad=\frac{\alpha^2}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^2}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^2}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_c\quad=\frac{\alpha}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_d\quad=\frac{1}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{1}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{1}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma.$$

In verification of the last three equations observe that they give

$$-\partial_b+(\beta+\gamma)\partial_c-\beta\gamma\partial_d=$$
$$\frac{\alpha^2-\alpha(\beta+\gamma)+\beta\gamma}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^2-\beta(\beta+\gamma)+\beta\gamma}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^2-\gamma(\beta+\gamma)+\beta\gamma}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma$$

that is $-\partial_b+(\beta+\gamma)\partial_c-\beta\gamma\partial_d=\partial_\alpha$: and similarly from the same three equations we deduce the values of ∂_β and ∂_γ; the three equations are thus equivalent to the foregoing three equations for ∂_α, ∂_β, ∂_γ.

As to the first equation, to avoid confusion with a root δ, I have written therein δ' (afterwards replaced by δ) to denote the degree of a function homogeneous in (a, b, c, d), upon which the symbols are supposed to operate; this is also the degree in the roots α, β, γ. The four equations give

$$-a(\partial_a-\delta')-b\partial_b-c\partial_c-d\partial_d=\frac{\alpha^3+b\alpha^2+c\alpha+d}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\text{etc.},=0,$$

since $\alpha^3+b\alpha^2+c\alpha+d=0$, $\beta^3+b\beta^2+c\beta+d=0$, $\gamma^3+b\gamma^2+c\gamma+d=0$. The equations thus give

$$a\partial_a+b\partial_b+c\partial_c+d\partial_d=\delta',$$

which is right, and the first equation is thus verified.

16. From the last three equations for ∂_b, ∂_c, ∂_d we deduce

$$-3\partial_b-2b\partial_c-c\partial_d=\frac{3\alpha^2+2b\alpha+c}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{3\beta^2+2b\beta+c}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{3\gamma^2+2b\gamma+c}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma.$$
$$=\partial_\alpha+\partial_\beta+\partial_\gamma,$$

a result more easily deducible from the first set of three equations for ∂_α, ∂_β, ∂_γ: but I have preferred to obtain it in this manner for the sake of the remark that it is a peculiarity of this combination of ∂_b, ∂_c, ∂_d that the coefficients of

$\partial_\alpha, \partial_\beta, \partial_\gamma$ become integral functions of the roots (in the actual case constants and $=1$): for a somewhat similar form

$$-(c\partial_b + d\partial_c), = \frac{c\alpha^2 + d\alpha}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha + \frac{c\beta^2 + d\beta}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta + \frac{c\gamma^2 + d\gamma}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\beta$$

the coefficients are fractional.

We at once have

$$\alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma = b\partial_b + 2c\partial_c + 3d\partial_d,$$

viz. these symbols operating upon a function of the roots of the degree ω, or what is the same thing, a function of the coefficients of the weight ω, are each of them equivalent to a constant factor ω.

Again we have

$$\begin{aligned}\alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma &= -(b^2 - 2c)\partial_b - (bc - 3d)\partial_c - bd\partial_d, \\ &= -b(b\partial_b + c\partial_c + d\partial_d) + 2c\partial_b + 3d\partial_c,\end{aligned}$$

or since $a\partial_a + b\partial_b + c\partial_c + d\partial_d = \delta'$ (if as before δ' is the degree of the function operated upon) and therefore $b\partial_b + c\partial_c + d\partial_d = \delta' - a\partial_a$ or say $= \delta' - \partial_a$, this is

$$\alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma = -b\delta' + b\partial_a + 2c\partial_b + 3d\partial_c,$$

so that we have here another form $-b\delta' + b\partial_a + 2c\partial_b + 3d\partial_c$, for which the coefficients of $\partial_\alpha, \partial_\beta, \partial_\gamma$ are integral functions of the roots.

17. In the case of four roots, the corresponding equations are

$$\begin{aligned}-b &= \alpha + \beta + \gamma + \delta, \\ +c &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta, \\ -d &= \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta, \\ +e &= \alpha\beta\gamma\delta,\end{aligned}$$

and we then have

$$\begin{aligned}\partial_\alpha &= -\partial_b + (\beta + \gamma + \delta)\partial_c - (\beta\gamma + \beta\delta + \gamma\delta)\partial_d + \beta\gamma\delta\partial_e, \\ \partial_\beta &= -\partial_b + (\gamma + \delta + \alpha)\partial_c - (\gamma\delta + \gamma\alpha + \delta\alpha)\partial_d + \gamma\delta\alpha\partial_e, \\ \partial_\gamma &= -\partial_b + (\delta + \alpha + \beta)\partial_c - (\delta\alpha + \delta\beta + \alpha\beta)\partial_d + \delta\alpha\beta\partial_e, \\ \partial_\delta &= -\partial_b + (\alpha + \beta + \gamma)\partial_c - (\alpha\beta + \alpha\gamma + \beta\gamma)\partial_d + \alpha\beta\gamma\partial_e,\end{aligned}$$

and the converse set of equations which for shortness I write in the form

$$-\partial_a + \delta', -\partial_b, -\partial_c, -\partial_d, -\partial_e =$$

$$\frac{\alpha^{4,3,2,1,0}}{\alpha-\beta.\alpha-\gamma.\alpha-\delta}\partial_\alpha + \frac{\beta^{4,3,2,1,0}}{\beta-\gamma.\beta-\delta.\beta-\alpha}\partial_\beta + \frac{\gamma^{4,3,2,1,0}}{\gamma-\delta.\gamma-\alpha.\gamma-\beta}\partial_\gamma + \frac{\delta^{4,3,2,0}}{\delta-\alpha.\delta-\beta.\delta-\gamma}\partial_\delta.$$

We have in like manner as in the former case

$$\begin{aligned} -4\partial_b - 3b\partial_c - 2c\partial_d - \ d\partial_e &= \partial_\alpha + \ \partial_\beta + \ \partial_\gamma + \ \partial_\delta, \\ b\partial_b + \ c\partial_c + \ d\partial_d + \ e\partial_e &= \alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma + \delta\partial_\delta, = \omega, \\ -b\delta' + b\partial_a + 2c\partial_b + 3c\partial_d + 4e\partial_d &= \alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma + \delta^2\partial_\delta; \end{aligned}$$

and similarly in the case of five or more roots.

18. In the case of σ' roots, I write $m = a_{\sigma'}$, and for shortness

$$\begin{aligned} \Theta_{\sigma'} &= \sigma'\partial_b + (\sigma'-1)b\partial_c \qquad \ldots\ldots + l\partial_m, \\ P &= \ b\partial_a + \qquad 2c\partial_b + 3d\partial_c \ldots\ldots + \sigma' m\partial_l, \end{aligned}$$

so that besides the equation $b\partial_b + c\partial_c \ldots\ldots + m\partial_m = S\alpha\partial_\alpha = \omega$, the foregoing investigations show that we have

$$\begin{aligned} \Theta_{\sigma'} &= - S\partial_\alpha, \\ P - b\delta &= \ S\alpha^2\partial_\alpha. \end{aligned}$$

The operand for these symbols is a symmetric function of the roots, which is thus also a function of the coefficients: it is of the degree ω in the roots, and consequently of the weight ω in the coefficients, and its degree in the coefficients is taken to be $= \delta$. It is sometimes convenient to represent this operand, quà function of the roots by Υ and quà function of the coefficients by U, so that we have in general $\Upsilon = U$. If Υ be a nonunitary function of the roots then we may say that $\Upsilon, = U$, is a nonunitariant.

19. I give some illustrations of the equation $\Theta_{\sigma'} = - S\partial_\alpha$. Suppose $\Upsilon = U = S\alpha^4 = E = -4e + 4bd + 2c^2 - 4b^2c + b^4$ (Table IV(b)); σ' must be $= 4$ at least and I take it to be 4 and 5 successively; we thus have

$$\begin{aligned} \Theta_4 &= 4\partial_b + 3b\partial_c + 2c\partial_d + \ d\partial_e, \\ \Theta_5 &= 5\partial_b + 4b\partial_c + 3c\partial_d + 2d\partial_e, \end{aligned}$$

omitting from Θ_5 the term $e\partial_f$ which is obviously inoperative. For any number whatever of roots we have $- S\partial_\alpha \,.\, S\alpha^4 = -4S\alpha^3 = -4(-3d + 3bc - b^3)$, $= 12d - 12bc + 4b^3$, and this should therefore be the value as well of $\Theta_4 E$ as of $\Theta_5 E$. The calculations may be arranged as follows:

$\Theta_4 E$

$4\ .$	$4d - 8bc + 4b^3$	$d + 16 \qquad -4$	12
$3b\ .$	$4c - 4b^2$	$bc - 32 + 12 + 8$	-12
$2c\ .$	$4b$	$b^3 + 16 - 12$	$+\ 4$
$d\ .$	-4		

$\Theta_5 E$

$5\ .$	$4d - 8bc + 4b^3$	$d - 20 \quad -8$	12
$4b\ .$	$4c - 4b^2$	$bc - 40 + 16 + 12$	-12
$3c\ .$	$4b$	$b^3 + 20 - 16$	$+\ 4$
$d\ .$	-4		

giving in each case the right result.

20. In the foregoing example $S\alpha^4$ was a nonunitary function of the roots, but I take the case of a unitary function. Suppose $\Upsilon = U = S\alpha^3\beta = BD = 4e - bd - 2c^2 + b^2c$. Here $-S\partial_\alpha \,.\, S\alpha^3\beta$ is not independent of the number of the roots; in the case of 4 roots we have $-S\partial_\alpha \,.\, S\alpha^3\beta = -3S\alpha^2\beta - 3S\alpha^3$, $= -3(3d - bc) - 3(-3d + 3bc - b^3)$, $= 0d - 6bc + 3b^3$; and in the case of 5 roots we have $-S\partial_\alpha \,.\, S\alpha^3\beta = -3S\alpha^2\beta - 4S\alpha^3$, $= -3(3d - bc) - 4(-3d + 3bc - b^3)$, $= 3d - 9bc + 4b^3$; and these should therefore be the values of $\Theta_4 BD$ and $\Theta_5 BD$ respectively. The calculations are

$\Theta_4 BD$

$4\ .$	$-\ d + 2bc$	$d - 4 \quad +4$	0
$3b\ .$	$-\ 4c + b^2$	$bc + 8 - 12 - 2$	-6
$2c\ .$	$-\ b$	$b^3 \quad + 3$	$+3$
$d\ .$	$+\ 4$		

$\Theta_5 BD$

$5\ .$	$-\ d + 2bc$	$d - 5 \quad +8$	$+3$
$4b\ .$	$-\ 4c + b^2$	$bc + 10 - 16 - 3$	-9
$3c\ .$	$-\ b$	$b^3 \quad + 4$	$+4$
$2d\ .$	$+\ b$		

giving in each case the correct result. We have $\Theta_5 - \Theta_4 = \partial_b + b\partial_c + c\partial_d + d\partial_e$, and the examples show that performing this operation on the nonunitariant $S\alpha^4$, $= E$ we obtain a result $= 0$; whereas for the unitary function $S\alpha^3\beta$, $= BD$, the result is not $= 0$.

21. Considering the question generally, I take the highest coefficient in U to be $q = a_\sigma$, (σ equal to or less than ω) or what is the same thing the extent of U to be $= \sigma$; this implies that σ' is at least $= \sigma$; and taking it to be first $= \sigma$, and then to be any number greater than σ, we have

$$\Theta_\sigma = -S\partial_\alpha, \quad \Theta_{\sigma'} = -S\partial_\alpha$$

where the function U operated upon by Θ_σ and $\Theta_{\sigma'}$ respectively is in each case the same function of the coefficients. It is easy to see that if Υ is a nonunitary function of the roots, then whatever be the number of the roots we have $S\partial_\alpha \,.\, \Upsilon =$ a determinate symmetric function of the roots, and consequently $=$ a determinate function of the coefficients. We thus have $\Theta_{\sigma'}U$ and $\Theta_\sigma U$ equal to each other; that is $(\Theta_{\sigma'} - \Theta_\sigma)U = 0$; we may write

$$\Theta_\sigma = \sigma\partial_b + (\sigma - 1)b\partial_c \ldots\ldots + \qquad\qquad p\partial_q,$$
$$\Theta_{\sigma'} = \sigma'\partial_b + (\sigma' - 1)b\partial_c \ldots\ldots + (\sigma' - \sigma + 1)p\partial_q,$$

for the subsequent terms of $\Theta_{\sigma'}$ as involving ∂_r, ∂_s, etc., are inoperative; hence writing

$$\Delta = \partial_b + b\partial_c + c\partial_b \ldots\ldots + p\partial_q,$$

or as we may more simply express it

$$\Delta = \partial_b + b\partial_c + c\partial_b + \ldots\ldots,$$

we have $\Theta_{\sigma'} - \Theta_\sigma = (\sigma' - \sigma)\Delta$, and consequently $\Delta U = 0$; Δ is thus an annihilator of any function U of the coefficients which is equal to a nonunitary function of the roots; or more shortly Δ is an annihilator of any nonunitariant.

22. Similarly from the two equations $\Theta_\sigma = -S\partial_\alpha$, and $\Theta_{\sigma'} = S\partial_\alpha$ regarded as operating upon a nonunitary function, we deduce $\sigma'\Theta_\sigma - \sigma\Theta_{\sigma'} = (\sigma - \sigma')S\partial_\alpha$: the left-hand side is here $= (\sigma - \sigma')\Delta_1$, if

$$\Delta_1 = b\partial_c + 2c\partial_b + 3c\partial_d \ldots\ldots + (\sigma - 1)p\partial_q,$$

or say

$$\Delta_1 = b\partial_c + 2c\partial_b + 3c\partial_d + \ldots\ldots$$

viz. we have $\Delta_1 = S\partial_\alpha$; for instance, if as before $\Omega = U = S\alpha^4 = -4e + 4bd + 2c^2 - 4b^2c + b^4$, then $(b\partial_c + 2c\partial_d + 3d\partial_e)(-4e + 4bd + 2c^2 - 4b^2c + b^4) = S\partial_\alpha \,.\, S\alpha^4$, $= 4S\alpha^3$, $= 4(-3d + 3bc - b^3)$, as can be at once verified. It is to be noticed, however, that $S\partial_\alpha$ operating upon a nonunitary function of the roots does not in every case give a nonunitary function; and thus successive operations with Δ_1 will not give a succession of nonunitariants.

23. I investigate the foregoing result in regard to Δ in a different manner; suppose for instance $\Upsilon = U$ is the nonunitary function $S\alpha^4$ of the roots, $(= -4e + 4bd + 2c^2 - 4b^2c + b^4)$. The number of roots is at least $= 4$, and I take it to be $= 4$, say the roots are $\alpha, \beta, \gamma, \delta$. Consider a fifth root θ, and let $\Upsilon_1 = U_1 = S\alpha^4$ be the like function for the five roots, we have $\Upsilon_1 = \Upsilon + \theta^4$, or

say $U_1 = U + \theta^4$. Write $-b_1, c_1, -d_1, e_1, -f_1$ for the symmetric functions of the five roots, U_1 will not involve f_1 and it will be the same function of b_1, c_1, d_1, e_1 that U is of b, c, d, e, say we have $U_1 = U(b_1, c_1, d_1, e_1)$. But we have $b_1 = b - \theta$, $c_1 = c - b\theta$, $d_1 = d - c\theta$, $e_1 = e - d\theta$; and thus the foregoing equation $U_1 = U + \theta^4$ becomes

$$U(b - \theta, c - b\theta, d - c\theta, e - d\theta) = U + \theta^4;$$

it is in fact easy to verify, that for the foregoing value of U, the terms in $\theta, \theta^2, \theta^3$ all vanish, and that the expression on the left hand becomes $= U + \theta^4$. But attending only to the term in θ, this is $= -\theta(\partial_b + b\partial_c + c\partial_d + d\partial_e)U$, $= -\theta\Delta U$; viz. this term vanishing we have $\Delta U = 0$, the result which was to be proved.

In the case of a unitary function, for instance $\Upsilon = U = S\alpha^3\beta$, here introducing the new root θ we have $U_1 = U + \theta S\alpha^3 + \theta^3 S\alpha$; or there is here a term in θ, and instead of $\Delta U = 0$, we have $\Delta U = S\alpha^3$, or the unitary function is not annihilated by Δ.

The foregoing investigation is really quite general, and establishes the conclusion that Δ is an annihilator of every nonunitariant.

It is to be noticed that Θ_σ and Δ are operators which leave each of them the degree unaltered but diminish the weight by unity: the operator $P - b\delta$, and another operator $\frac{1}{2}Q - b\omega$ which will be considered, increase each of them the degree by unity and also the weight by unity.

24. Coming now to the equation

$$P - b\delta = S\alpha^2\partial_a$$

it is to be remarked that if $\sigma' = \sigma$, the expression for P ends in $q\partial_p$, where as before $q = a_\sigma$ is the highest coefficient in the operand; since the operand thus contains q, the next succeeding term in $r\partial_q$ would be not inoperative, and in order to include it in the expression of P we may take $\sigma' = \sigma + 1$; we thus have

$$P = b\partial_a + 2c\partial_b + 3d\partial_c \ldots\ldots + (\sigma + 1)r\partial_q,$$

or as we may more simply write it

$$P = b\partial_a + 2c\partial_b + 3d\partial_d + \ldots\ldots$$

the operation thus increases the extent by unity. The symbol $S\alpha^2\partial_a$ operating upon a symmetric function of the roots, gives, whatever may be the number of

roots, the same symmetric function of the roots: and we see further that operating upon a nonunitary function it gives a nonunitary function of the roots. Hence $P - b\delta$ operating upon a nonunitariant gives a nonunitariant. I give an example.

25. Suppose as before $\Upsilon = U = Sa^4 = E = -4e + 4bd + 2c^2 - 4b^2c + b^4$, here $\delta = 4$ and therefore $P - b\delta = b\partial_a + 2c\partial_b + 3d\partial_c + 4e\partial_d + 5f\partial_e - 4b$. We have $Sa^2\partial_a \,.\, Sa^4 = 4Sa^5$, $= 4(-5f + 5be + 5cd - 5b^2d - 5bc^2 + 5b^3c - b^5)$, and this should therefore be the result of the operation $P - b\delta$: the calculation is

$b\,.$	$-12c + 8bd + 4c^2 - 4b^2c$	f					-20		-20
$2c\,.$	$4d - 8bc + 4b^3$	be	-12			$+16$		$+16$	$+20$
$3d\,.$	$4c - 4b^2$	cd		$+8$	$+12$				$+20$
$4e\,.$	$4b$	b^2d	$+8$		-12			-16	-20
$5f\,.$	-4	bc^2	$+4$	-16				-8	-20
$-4b\,.$	$-4e + 4bd + 2c^2 - 4b^2c + b^4$	b^3c	-4	$+8$				$+16$	$+20$
		b^5						-4	-4

which is the right result.

We have seen that every nonunitariant is annihilated by Δ; it at once appears that conversely every function of the coefficients which is annihilated by Δ is a nonunitariant: it is in fact a symmetric function of the roots, and unless it were a nonunitary function of the roots it would not be annihilated by Δ. Nonunitariants are analogous to seminvariants; the precise relation between them will be shown further on.

26. We can by an investigation similar to that for seminvariants, show that $P - b\delta$ operating upon a nonunitariant gives a nonunitariant. In fact considering the two operations Δ and $P - b\delta$, we have

$$\Delta(P - b\delta)\dagger = \Delta(P - b\delta) + \Delta \,.\, (P - b\delta)\,,$$

the meaning being that if upon any operand U we perform first the operation $P - b\delta$ and then the operation Δ, this is equivalent to operating on U with the sum of the two operations $\Delta(P - b\delta)$, and $\Delta \,.\, P - b\delta$, the first of these symbols denoting the mere algebraical product of Δ and $P - b\delta$, the second of them the result of the operation Δ performed upon $P - b\delta$. We have similarly $(P - b\delta)\Delta\dagger = (P - b\delta)\Delta + (P - b\delta) \,.\, \Delta$.

Hence observing that $\Delta(P-b\delta)$ and $(P-b\delta)\Delta$ are equal to each other, and subtracting, we have

$$\Delta(P-b\delta)\dagger-(P-b\delta)\Delta\dagger=\Delta\,.\,(P-b\delta)-(P-b\delta)\,.\,\Delta.$$

But from the values

$$\Delta=a\partial_b+b\partial_c+c\partial_d+\ldots\,,$$

and

$$P-b\delta=b\partial_a+c\partial_b+d\partial_c+\ldots-b\delta,$$

we find

$$\begin{aligned}\Delta\,.\,(P-b\delta)&=a\partial_a+2b\partial_b+3c\partial_c+\ldots-\delta,\\(P-b\delta)\,.\,\Delta&=\qquad\quad b\partial_b+2c\partial_c+\ldots,\end{aligned}$$

and thence

$$\Delta\,.\,(P-b\delta)-(P-b\delta)\,.\,\Delta=a\partial_a+b\partial_b+c\partial_c\ldots-\delta,=0$$

since δ is the degree in the coefficients. Hence writing down the operand U,

$$\Delta\,.\,(P-b\delta)U-(P-b\delta)\,.\,\Delta U=0$$

where for greater clearness I have inserted the dots, to show that Δ operates on $(P-b\delta)\,U$, and $(P-b\delta)$ on ΔU. Taking U to be a nonunitariant we have $\Delta U=0$, and this being so the equation gives $\Delta\,.\,(P-b\delta)\,U=0$, viz. this shows that $(P-b\delta)\,U$ is a nonunitariant.

27. There is another symbol $\frac{1}{2}Q-b\omega$, which is precisely analogous to $P-b\delta$, viz. operating upon a nonunitariant, it gives a nonunitariant: ω is as before the weight of the function operated upon, and the expression of Q is

$$\tfrac{1}{2}Q=c\partial_b+3d\partial_c+6e\partial_d+\ldots+\tfrac{1}{2}\sigma(\sigma+1)r\partial_q,$$

or say

$$\tfrac{1}{2}Q=c\partial_b+3d\partial_c+6e\partial_d+\ldots$$

The proof is exactly similar, viz. we have to show that

$$\Delta\,.\,(\tfrac{1}{2}Q-b\omega)-(\tfrac{1}{2}Q-b\omega)\,.\,\Delta=0.$$

We have

$$\begin{aligned}\Delta\,.\,(\tfrac{1}{2}Q-b\omega)&=b\partial_b+3c\partial_c+6d\partial_d\ldots-\omega\\(\tfrac{1}{2}Q-b\omega)\,.\,\Delta&=\qquad\quad c\partial_c+3d\partial_d\ldots\end{aligned}$$

and the difference of the two expressions is

$$b\partial_b+2c\partial_c+3d\partial_d+\ldots-\omega,=0$$

since ω is the weight of the function operated upon. Hence as before if U be a nonunitariant and therefore $\Delta U=0$, we have $\Delta\,.\,(\frac{1}{2}Q-b\omega)\,U=0$, that is $(\frac{1}{2}Q-b\omega)\,U$ is also a nonunitariant.

28. The symbol $\frac{1}{2}Q-b\omega$ has no simple expression in terms of ∂_α, ∂_β, ∂_γ, ... and the form varies with the number of the roots: thus for 3 roots it is

$$=-\left\{\left(\frac{c\alpha^2+3d\alpha}{\alpha-\beta\,.\,\alpha-\gamma}+b\alpha\right)\partial_\alpha+\text{etc.}\right\},$$

for 4 roots it is

$$=-\left\{\left(\frac{c\alpha^3+3d\alpha^2+6e\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta}+b\alpha\right)\partial_\alpha+\text{etc.}\right\},$$

for 5 roots it is

$$=-\left\{\left(\frac{c\alpha^4+3d\alpha^3+6e\alpha^2+10f\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\varepsilon}+b\alpha\right)\partial_\alpha+\text{etc.}\right\}$$

and so on. It is not easy to find the effect of such a symbol upon a given symmetric function of the roots, nor in particular when the function is nonunitary is it easy to show generally that the result is nonunitary.

It is to be remarked that if the function operated upon is of the degree δ in the roots, then we must for $\frac{1}{2}Q-b\omega$ take the expression with $\delta+1$ roots; for instance, if the function be of the degree 5 in the roots, then quà function of the coefficients this contains f, and it must be operated on with $\frac{1}{2}Q-b\omega$, $=c\partial_b+3d\partial_c+6c\partial_d+10f\partial_e+15g\partial f-\omega b$, viz. this expression, as containing g, gives the 6-root expression for $\frac{1}{2}Q-\omega b$.

29. Suppose for instance the function operated upon is $F=S\alpha^5$; here taking the 6-root expression this gives

$$-5\left\{\left(\frac{c\alpha^5+3d\alpha^4+6e\alpha^3+10f\alpha^2+15g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\varepsilon\,.\,\alpha-\zeta}+b\alpha\right)\alpha^4+\text{etc.}\right\}$$

or omitting for the moment the outside factor -5, the expression in $\{\}$ is easily seen to be

$$=cH_4+3dH_3+6eH_2+10fH_1+15g+bS\alpha^5,$$

where H_4, H_3, H_2, H_1 denote the homogeneous functions of the degrees 4, 3, 2, 1 respectively: the values of these are obtained by adding together all

the lines of the Table IV(b), all the lines of the Table III(b), etc.: the terms exclusive of $6Sa^5$ thus are

$$\begin{aligned}
&\quad c(-e+2bd+c^2-3b^2c+b^4)\\
&+\ 3d(-d+2bc-b^3)\\
&+\ 6e(-c+\ b^2)\\
&+10f(-b)\\
&+15g\,. \qquad\quad 1
\end{aligned}$$

and these are $= Sa^5\beta + Sa^4\beta^2 + Sa^3\beta^3$, as appears by the following calculation:

							$Sa^5\beta$	$Sa^4\beta^2$	$Sa^3\beta^3$	
g					$+15$	$+15$	$+6$	$+6$	$+3$	$+15$
bf				-10		-10	-1	-6	-3	-10
ce	-1		-6			-7	-6	$+2$	-3	-7
b^2e			$+6$			$+6$	$+1$	$+2$	$+3$	$+6$
d^2		-3				-3	-3	-3	$+3$	-3
bcd	$+2$	$+6$				$+8$	$+7$	$+4$	-3	$+8$
b^3d		-3				-3	-1	-2	0	-3
c^3	$+1$					$+1$	$+2$	-2	$+1$	$+1$
b^2c^2	-3					-3	-4	$+1$		-3
b^4c	$+1$					$+1$	$+1$			$+1$
b^6										

The omitted term bSa^5, that is $-Sa\,.\,Sa^5$, is $-Sa^6 - Sa^5\beta$; the addition hereof destroys therefore the nonunitary term $Sa^5\beta$, and thus the required expression, restoring the omitted factor -5 is $-5(-Sa^6+Sa^4\beta^2+Sa^3\beta^3)$, or say $=5G-5CE-5D^2$, a nonunitary form: this then should be the result of the operation $\frac{1}{2}Q-b\omega$, $=c\partial_b+3d\partial_c+6e\partial_d+10f\partial_e+15g\partial_f-5b$ performed upon $Sa^5=F=-5f+5be+5cd-5b^2d-5bc^2+5b^3c-b^5$. Performing the calculation so as to omit on each side a factor 5, it is to be shown that $G-CE-D^2$ is $=$

$$\begin{aligned}
&\quad c(\,e-2bd-c^2+3b^2c-b^4)\\
&+\ 3d(\,d-2bc+b^3)\\
&+\ 6e(\,c-b^2)\\
&+10f(b)\\
&+15g(-1)\\
&-\ 5b(-f+be+cd-b^2d-bc^2+b^3c-\tfrac{1}{5}b^5).
\end{aligned}$$

Collecting the terms, and comparing the result with the expression for $G-CE-D^2$, we have

								$G-CE-D^2$		
g					-15		-15	-6	-6	-3
bf				$+10$		$+5$	$+15$	$+6$	$+6$	$+3$
ce	$+1$		$+6$				$+7$	$+6$	-2	$+3$
b^2e			-6			-5	-11	-6	-2	-3
d^2		$+3$					$+3$	$+3$	$+3$	-3
bcd	-2	-6				-5	-13	-12	-4	$+3$
b^3d		$+3$				$+5$	$+8$	$+6$	$+2$	0
c^3	-1						-1	-2	$+2$	-1
b^2c^2	$+3$					$+5$	$+8$	$+9$	-1	
b^4c	-1					-5	-6	-6		
b^6						$+1$	$+1$	$+1$		

and the two expressions are thus identical.

30. Suppose again, 6 roots as before, that the function operated upon is $S\alpha^3\beta^2$; we find $\partial_\alpha S\alpha^3\beta^2=3\alpha^2S\alpha^2+2\alpha S\alpha^3-5\alpha^4$, and the general term is

$$-3\Big(\frac{c\alpha^5+3d\alpha^4+6e\alpha^3+10f\alpha^2+16g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\varepsilon\,.\,\alpha-\zeta}+b\alpha\Big)\alpha^2\,.\,S\alpha^2$$

$$-2\Big(\frac{c\alpha^5+3d\alpha^4+6e\alpha^3+10f\alpha^2+15g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\varepsilon\,.\,\alpha-\zeta}+b\alpha\Big)\alpha\,.\,S\alpha^3$$

$$+5\Big(\frac{c\alpha^5+3d\alpha^4+6e\alpha^3+10f\alpha^2+15g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\varepsilon\,.\,\alpha-\zeta}+b\alpha\Big)\alpha^4$$

This gives

$$\begin{aligned}&-3(CH_2+3dH_1+6e+bS\alpha^3)S\alpha^2\\&-2(CH_1+3d\qquad\quad\;+bS\alpha^2)S\alpha^3\\&+5(CH_4+3dH_3+6eH_2+10fH_1+15g+bS\alpha^5)\end{aligned}$$

which is found to be

$$\begin{aligned}=&-3(BD+C^2\;+bS\alpha^3)S\alpha^2\\&-2(BC\qquad+bS\alpha^2)S\alpha^3\\&+5(BF+CE+D^2+bS\alpha^5).\end{aligned}$$

Here $bS\alpha^3=-S\alpha S\alpha^3=-S\alpha^4-S\alpha^3\beta,=-E-BD$; $bS\alpha^2=-S\alpha S\alpha^2=-S\alpha^3-S\alpha^2\beta=-D-BC$; and $bS\alpha^5=-S\alpha S\alpha^5=-S\alpha^6-S\alpha^5\beta,=-G-BF$; the expression thus is

$$
\begin{array}{llll}
= -3(-E + C^2) \,.\, C & & \text{that is } -3(-Sa^4 + Sa^2\beta^2)Sa^2 & \\
\;\;\, -2(-D \qquad) \,.\, D & & \qquad\quad\;\; -2(-Sa^3 \qquad\quad)Sa^3 & \\
\;\;\, +5(-G + CE + D^2) & & \qquad\quad\;\; +5(-Sa^6 + Sa^4\beta^2 + Sa^3\beta^3). &
\end{array}
$$

Here $Sa^2Sa^4 = Sa^6 + Sa^4\beta^2, \; = G + CE$; $Sa^3Sa^3 = Sa^6 + 2Sa^3\beta^3, \; = G + 2D^2$; $Sa^2Sa^2\beta^2 = Sa^4\beta^2 + 3Sa^2\beta^2\gamma^2, \; = CE + 3C^3$; and the whole is

$$
\begin{array}{l}
-3\{-G - CE + (CE + 3C^2)\} \\
-2(-G - 2D^2) \\
+5(-G + CE + D^2)
\end{array}
$$

which is $= 5CE + 9D^2 - 9C^3$ (a nonunitary form). This then should be the value of $\frac{1}{2}Q - b\omega, \; = c\partial_b + 3d\partial_c + 6e\partial_d + 10f\partial_e + 15g\partial_f - 5b$ operating upon $Sa^3\beta^2, \; = CD = 5f - 5be + cd + 2b^2d - bc^2$.

31. There is for nonunitariants a theorem which is a much more simple form than the transformation of it afterwards obtained for seminvariants: viz. for any nonunitariant we have $\Delta U = 0 = (\partial_b + b\partial_c + c\partial_d + \ldots.)U$; attending only to the portion U' of U which is of the highest degree, it is clear that we have $(b\partial_c + c\partial_d + \ldots.)U' = 0$, and if we herein diminish the letters then $(\partial_b + b\partial_c + \ldots.)U'' = 0$, where U'' is what U' becomes by a diminution of the letters; that is U'' is a nonunitariant, viz. in any seminvariant, the terms of highest degree U' are obtained from a nonunitariant U'' by a mere augmentation of the letters: e. g. $2e - 2bd + c^2$ is a nonunitariant weight 4; augmenting the letters we have $2bf - 2ce + d^2$ which with a change of sign is the portion of highest degree of the nonunitariant $2g - 2bf + 2ce - d^2$.

The MacMahon Form of Equation. Article Nos. 32 *to* 34.

32. The equation connecting the coefficients and the roots is here taken to be

$$1 + \frac{b}{1}x + \frac{c}{1\,.\,2}x^2 + \frac{d}{1\,.\,2\,.\,3}x^3 + \ldots. = 1 - \alpha x \,.\, 1 - \beta x \,.\, 1 - \gamma x \ldots.$$

As to this it may be remarked that if we had started with a form of the n^{th} order with binomial coefficients,

$$1 + \frac{n}{1}bx + \frac{n\,.\,n-1}{1\,.\,2}cx^2 + \frac{n\,.\,n-1\,.\,n-1}{1\,.\,2\,.\,3}dx^3 + \ldots.$$
$$= 1 - \alpha x \,.\, 1 - \beta x \,.\, 1 - \gamma x \ldots. \;(n \text{ factors}),$$

then writing herein $\frac{x}{n}$ for x, and also $n\alpha, n\beta, n\gamma, \ldots,$ for $\alpha, \beta, \gamma, \ldots$ and putting ultimately $n = \infty$ we have the form in question.

We pass from the ordinary form to the MacMahon form, by writing for $b, c, d, e, \ldots, \frac{b}{1}, \frac{c}{1.2}, \frac{d}{1.2.3}, \frac{e}{1.2.3.4}, \ldots$ or say $b, \frac{c}{2}, \frac{d}{6}, \frac{e}{24}, \frac{f}{120}, \frac{g}{720}, \ldots$ All the results obtained for the ordinary form will, after making therein this change, apply to the new form. We thus find

$$\Theta_\sigma = \sigma\partial_b + (\sigma - 1)2b\partial_c + (\sigma - 2)3c\partial_d \ldots + \qquad 1\sigma p\partial_q,$$
$$\Theta_{\sigma'} = \sigma'\partial_b + (\sigma' - 1)2b\partial_c + (\sigma' - 2)3c\partial_d \ldots + (\sigma' - \sigma + 1)\sigma p\partial_q,$$

$\Theta_\sigma - \Theta_{\sigma'} = (\sigma' - \sigma)\Delta$ where

$$\Delta = \partial_b + 2b\partial_c + 3c\partial_d \ldots + \sigma p\partial_q, \text{ or say}$$
$$= \partial_b + 2b\partial_c + 3c\partial_d + \ldots$$

Also

$$P = b\partial_a + c\partial_b + d\partial_c \ \ldots + r\partial_q, \text{ or say}$$
$$= b\partial_a + c\partial_b + d\partial_c \ + \ldots,$$
$$Q = \qquad c\partial_b + 2d\partial_c \ldots + \sigma r\partial_q, \text{ or say}$$
$$= \qquad c\partial_b + 2d\partial_c + \ldots$$

The change $\alpha, \beta, \gamma, \ldots$ into $n\alpha, n\beta, n\gamma, \ldots$ would change $S\partial_\alpha, S\alpha\partial_\alpha, S\alpha^2\partial_\alpha$ into $n^{-1}S\partial_\alpha, S\alpha\partial_\alpha, nS\alpha^2\partial_\alpha$ respectively ($n = \infty$): but this change is in fact compensated for by the introduction into the formulæ of the binomial coefficients as above; it is $-S\alpha, S\alpha\beta, \ldots$ not $-nS\alpha, n^2S\alpha\beta, \ldots$ which are equal to $b, \frac{1}{2}c, \ldots$; and the conclusion is that we have to retain without alteration the symbols $S\partial_\alpha, S\alpha\partial_\alpha, S\alpha^2\partial_\alpha$: thus in the new form as in the old one we have $\Theta_4 S\alpha^4 = -S\partial_\alpha \,.\, S\alpha^4 = -4S\alpha^3$, see the example *ante* No. 23.

33. In the new form, a nonunitariant is annihilated by the operator $\Delta, = \partial_b + 2b\partial_c + 3c\partial_d + \ldots$, and conversely any function annihilated by Δ is a nonunitariant; comparing herewith the subsequent theory of seminvariants, this is in fact the theorem that a nonunitariant is the same thing as a seminvariant; or to state this more explicitly: for the MacMahon form of equation, a function of the coefficients which is a nonunitary symmetric function of the roots is a seminvariant.

I consider for instance the Table VI(b), but attend only to the nonunitary portions thereof, viz. the lines G, CE, D^2, C^3: I convert these into columns, at

the same time changing the arrangement of the headings g, bf, ce, etc., from CO to AO: and then making the foregoing change b, c, d, e, f, g into $b, \frac{c}{2}, \frac{d}{6}, \frac{e}{24}, \frac{f}{120}, \frac{g}{720}$, but to avoid fractions multiplying the whole by 720, I form the table

	$\div 720$ $\Vert$	C^3	D^2	CE	G
1	g	-2	$+3$	$+6$	-6
6	bf	$+2$	-3	-6	$+6$
15	ce	-2	-3	$+2$	$+6$
20	d^2	$+1$	$+3$	-3	$+3$
30	b^2e		$+3$	$+2$	-6
60	bcd		-3	$+4$	-12
90	c^3		$+1$	-2	-2
120	b^3d			-2	$+6$
180	b^2c^2			$+1$	$+9$
360	b^4c				-6
720	b^6				$+1$
		$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

which is to be read according to the columns: and observe that the outside left-hand numbers are to be multiplied into the numbers of each column: thus the first column is to be read $C^3 = S\alpha^2\beta^2\gamma^2 = \frac{1}{720}(-\,2bf + 12ce - 30ce + 20d^2)$: the second column is to be read $D^2 = S\alpha^3\beta^3 = \frac{1}{720}(3g - 18bf \ldots\ldots + 90c^3)$, and so on.

By what precedes the columns are seminvariants,—as afterwards explained, "blunt" seminvariants; and they are named as such by the outside bottom line of symbols with a [ ]; viz. $[d^2] = (-\,2g + 12bf - 30ce + 20d^2)$, $[c^3] = (3g - 18bf \ldots\ldots + 90c^3)$, etc., where it will be observed that the symbol within the [ ] is in fact the power-ender which is in AO the lowest term of the column; and further that this is also the conjugate of the capital letter symbol at the head of the column.

The (b) Tables I to X, with only the change $b, c, d, e, \ldots\ldots$ into $b, \frac{c}{2}, \frac{d}{6}, \frac{e}{24}, \ldots\ldots$ are given in my paper, Tables of the Symmetric Functions of the Roots

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to the degree 10, for the form $1 + bx + \frac{cx^2}{1\,.\,2} + \ldots = (1-\alpha x)(1-\beta x)(1-\gamma x)\ldots$ Amer. Math. Jour., t. VII (1885), pp. 47–56.

34. By what precedes it appears that $P - b\delta$ operating on a seminvariant gives a seminvariant, and that $Q - 2b\omega$ operating on a seminvariant gives a seminvariant: these operators will be further considered in the development of the theory of seminvariants. We see further that $\frac{1}{2}\Delta, = b\partial_c + 3c\partial_d + 6d\partial_e + \ldots$ operating on a seminvariant gives sometimes but not always a seminvariant, e. g. $(b\partial_c + 3c\partial_d + 6d\partial_e)(e - 4bd - 3c^2 + 12b^2c - 6b^4) = 6(d - 3bc + 2b^3)$.

Seminvariants—the I-and-F Problem, and Solution by Square Diagrams. Article Nos. 35 *to* 47.

35. Writing

$$\begin{aligned}
1 &= 1,\\
b_1 &= b + \theta,\\
c_1 &= c + 2b\theta + \theta^2,\\
d_1 &= d + 3c\theta + 3b\theta^2 + \theta^3,\\
e_1 &= e + 4d\theta + 6c\theta^2 + 4b\theta^3 + \theta^4,
\end{aligned}$$

etc.,

then there are functions of the unsuffixed letters which remain unaltered if for these we substitute the suffixed letters: any such function is termed a seminvariant. We have for instance

$$\begin{array}{rll}
c_1 = & c + 2b\theta + \theta^2, & \text{i. e.} \quad c_1 - b_1^2 = c - b^2,\\
-b_1^2 = & -b^2 - 2b\theta - \theta^2, & \\
d_1 = & d + 3c\theta + 3b\theta^2, & d_1 - 3b_1c_1 + 2b_1^3 = d - 3bc + 2b^3,\\
-3b_1c_1 = & -3bc - 6b^2\theta - 3b\theta^2, & \\
 & -3c\theta - 6b\theta^2 - 3\theta^3, & \\
+2b_1^3 = & 2b^3 + 6b^2\theta + 6b\theta^2 + 2\theta^3, &
\end{array}$$

and thus $c - b^2$, $d - 3bc + 2b^3$ are seminvariants; they are in fact the first and second terms of the series

$$
\begin{aligned}
&c - b^2,\\
&d - 3bc + \ \ 2b^3,\\
&e - 4bd + \ \ 6b^2c - \ \ 3b^4,\\
&f - 5bc + 10b^2d - 10b^3c + \ \ 4b^5,\\
&g - 6bf + 15b^2e - 20b^3d + 15b^4c - 5b^6,\\
&\ \vdots
\end{aligned}
$$

where the law is obvious; the numbers in each line are binomial coefficients except the last number, which is the next binomial coefficient diminished by unity. The successive terms are in fact what $c_1, d_1, e_1, f_1, g_1, \ldots$ become upon writing therein $\theta = -b$.

36. Any rational and integral function of these forms is a seminvariant, and it is to be observed that we can form functions for which (by the destruction of terms of a higher degree) there is a diminution of degree; for instance $(e - 4bd + 6b^2c - 3b^4) + 3(c - b^2)^2$ gives a seminvariant $e - 4bd + 3c^2$.

It is important to remark that a seminvariant is completely determined by its nonunitary terms, thus for $e - 4bd + 3c^2$, the nonunitary terms are $e + 3c^2$, and for this writing $e_1 + 3c_1^2$, and for e_1, c_1 substituting their above values for $\theta = -b$, we reproduce the original value $e - 4bd + 3c^2$.

37. It is at once seen that a seminvariant is reduced to zero by the operation $\Delta, = \partial_b + 2b\partial_c + 3c\partial_d + \ldots$, or say that Δ is an annihilator of a seminvariant; in fact, if in any function of $b, c, d, \ldots$ we write for these the suffixed letters $b_1, c_1, d_1, \ldots$ then the coefficient of θ herein is at once found by operating on the function of $(b, c, d, \ldots)$ with Δ, and therefore in the case of a seminvariant the result of this operation must be $= 0$. And conversely every function of $(b, c, d, \ldots)$ which is reduced to zero by the operation Δ is a seminvariant.

38. For a given weight the number of seminvariants is equal to the excess of the number of terms of that weight above the number of terms of the next preceding weight, or what is the same thing it is equal to the number of power-enders of the given weight. More definitely, considering the terms of a seminvariant as arranged in AO, we have seminvariants the finals whereof are the several power-enders of the given weight: and we arrange the seminvariants *inter se* by taking these power-enders in AO: thus for the weight 6 we have seminvariants $[d^2]$, $[c^3]$, $[b^2c^2]$, $[b^6]$ ending in these terms respectively. We may

if we please consider all these seminvariants as beginning with g, or say the forms may be taken to be $g(\text{ao})d^2$, $g(\text{ao})c^3$, $g(\text{ao})b^2c^2$, $g(\text{ao})b^6$. Such forms are in fact furnished by the MacMahon equation: viz. up to the weight 6 we thus have

÷2 ‖	C
$1c$	-2
$2b^2$	$+1$
	$[b^2]$

÷6 ‖	D
$1d$	-3
$3bc$	$+3$
$6b^3$	-1
	$[b^3]$

÷24 ‖	C^2	E
$1e$	$+2$	-4
$4bd$	-2	$+4$
$6c^2$	$+1$	$+2$
$12b^2c$		-4
$24b^4$		$+1$
	$[c^2]$	$[b^4]$

÷120 ‖	CD	F
$1f$	$+5$	-5
$5be$	-5	$+5$
$10cd$	$+1$	$+5$
$20b^2d$	$+2$	-5
$30bc^2$	-1	-5
$60b^3c$		$+5$
$120b^5$		-1
	$[bc^2]$	$[b^5]$

÷720 ‖	C^3	D^2	CE	G
$1g$	-2	$+3$	$+6$	-6
$6bf$	$+2$	-3	-6	$+6$
$15ce$	-2	-3	$+2$	$+6$
$20d^2$	$+1$	$+3$	-3	$+3$
$30b^2e$		$+3$	$+2$	-6
$60bcd$		-3	$+4$	-12
$90c^3$		$+1$	-2	-2
$120b^3d$			-2	$+6$
$180b^2c$			$+1$	$+9$
$360b^4c$				-6
$720b^6$				$+1$
	$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

for the present purpose read for instance

$$[d^2] = -\ 2g + 12bf - 30ce + 20d^2,$$
$$[c^3] = \ 3g - 18bf - 45ce + 60d^2 + 90b^2e - 180bcd + 90c^3,$$
etc.

and I say that $[d^2]$, $[c^3]$, $[b^2c^2]$, $[b^6]$ are "specific" when they are regarded as standing for these tabulated functions; but in general I take them to be "indefinite," that is I regard them as denoting (as above) any seminvariants ending in d^2, c^3, b^2c^2, b^6 respectively.

39. The seminvariant $[d^2]$ is of the form $(g \propto d^2)$, including those terms which are in CO not superior to g and in AO not inferior to d^2: by a combination of $[d^2]$ and $[c^3]$ we obtain a seminvariant $(ce \propto c^3)$ containing terms which are in CO not superior to ce and in AO not inferior to c^3: similarly from $[d^2]$, $[c^3]$, $[b^2c^2]$ we obtain a seminvariant $(d^2 \propto b^2c^2)$; and from the four forms a seminvariant $(c^3 \propto b^6)$: these four seminvariants

	$(g \infty d^2)$	$(ce \infty c^3)$	$(d^2 \infty b^2c^2)$	$(c^3 \infty b^6)$
g	$+1$			
bf	-6			
ce	$+15$	$+1$		
d^2	-10	-1	$+1$	
b^2e		-1		
bcd		$+2$	-6	
c^3		-1	$+4$	$+1$
b^3d			$+4$	
b^2c^2			-3	-3
b^4c				$+3$
b^6				-1

are said to be "sharp" seminvariants: viz. considering the final as given, a sharp seminvariant is one having an initial which is in CO as low as possible; or considering the final as given, it is one having a final which is in AO as high as possible. A seminvariant which is not sharp is said to be "blunt."

40. The sharp seminvariants are in general designated as above, $(g \infty d^2)$, etc.: but it is sometimes convenient to give the numerical coefficients of the initial and final terms respectively: as to this it is to be noticed that the coefficient of the initial term is in most cases, but not always $= 1$, — we might of course take it to be always $= 1$, but we should then in the excepted cases have fractional coefficients, and it is better to avoid this by giving a proper value to the numerical coefficient of the initial term; the numerical coefficient of the final term is in general different from ± 1, and it is not in general a multiple of the numerical coefficient of the initial term. As an instance take $dh \infty b^2e^2$, the more complete expression of which is $4dh \infty -35b^2e^2$. The sharp seminvariants up to the weight 12 are designated in this more complete form in the table *post* No. 62.

41. In the calculation of the sharp seminvariants by elimination as above, it will be noticed how unitary terms disappear: thus in combining $[d^2]$ and $[c^3]$ so as to get rid of g, the term bf disappears of itself, and we have as above the form $(ce \infty c^3)$ beginning with the nonunitary term ce. We may in fact write $b = 0$, we thus have

$$[d^2] = -2g - 30ce + 20d^2,$$
$$[c^3] = 3g - 45ce + 60d^2 + 90c^3,$$

giving $3[d^2]+2[c^3]=-180(ce-d^2-c^3)$, and then $ce-d^2-c^3$, putting therein for c, d, e the values $c-b^2$, $d-3bc+2b^3$, $e-4bd+6b^2c-3b^4$, gives the complete value *ut supra*, $ce-d^2-b^2e+2bcd-c^3$, and we thus see *à priori* that this contains no term bf, but in fact begins with ce. And in carrying out this process for any higher given weight, it is proper also to arrange the non-unitary terms not in AO but in CO, and then in each case beginning with the terms highest in CO and eliminating as many as possible of these terms we obtain the sharp seminvariant. Consider for instance the weight 12: taking the finals in AO we have here $(m \infty g^2)$, $(m \infty cf^2)$, $(m \infty e^3)$, $(m \infty b^2f^2)$, ... the initials in CO are m, ck, dj, ei, ... and it might at first sight appear that the foregoing process of elimination would lead to the forms $(m \infty g^2)$, $(ck \infty cf^2)$, $(dj \infty e^3)$, $(ei \infty b^2f^2)$, ...; we in fact have the form $(m \infty g^2)$; and if from $(m \infty g^2)$ and $(m \infty cf^2)$ we eliminate m, we obtain the form $(ck \infty cf^2)$; but we cannot have a form $(dj \infty e^3)$ (for a form beginning with dj is of necessity of the degree 4 at least); what happens is that when from $(m \infty g^2)$, $(m \infty cf^2)$ and $(m \infty e^3)$ we eliminate m and ck, the next term dj disappears of itself, and (the following term ei not disappearing) the resulting form is $(ei \infty e^3)$: to obtain a form beginning with dj we must use the fourth form $(m \infty b^2f^2)$, and we thence obtain $(dj \infty b^2f^2)$. Arranging the initials in CO and the finals in AO we thus have

$$\begin{array}{lcl} m & \text{———} & g^2 \\ ck & \text{———} & cf^2 \\ dj & & e^3 \\ ei & & b^2f^2 \end{array}$$

and then arranging the finals in AO we thus have the sharp seminvariants $m \infty g^2$, $ck \infty cf^2$, $ei \infty e^3$, $dj \infty b^2f^2$, ...; these are the results given by the MacMahon linkage as will be explained further on, but I will first approach the question from a different side.

42. It has been seen that we have $\Delta, = \partial_b + 2b\partial_c + 3c\partial_d + \ldots.$ as the annihilator of a seminvariant. Considering in the first place the entire set of terms, say for the weight 6, $g(\text{ao})b^6$, we assume for a seminvariant the sum of these each multiplied by an arbitrary coefficient, the number of coefficients is equal to the number of terms of $g(\text{ao})b^6$. Operating with Δ we obtain a function of the next inferior weight 5, containing all the terms of $Dg(\text{ao})b^6$, that is of $f(\text{ao})b^5$, each

term multiplied by a linear function (with mere numerical factors) of the arbitrary coefficients: the expression thus obtained must be identically $= 0$; and we thus find between the arbitrary coefficients a number of linear relations equal to the number of terms $f(\mathrm{ao})b^5$: these relations are independent; for it is only on the supposition that they are so, that the number of coefficients which remain arbitrary will be $11 - 7, = 4$, agreeing with the number of the seminvariants $[d^2]$, $[c^3]$, $[b^2c^2]$, $[b^6]$; whereas if the relations were not independent there would be a larger number of seminvariants.

But if instead of the whole set $g(\mathrm{ao})b^6$ we consider a set $(g \infty d^2)$ or say $(ce \infty c^3)$ and assume for a seminvariant the sum of these terms each multiplied by an arbitrary coefficient, then operating as before with Δ we obtain between the arbitrary coefficients a number of relations equal to that of the terms $D(ce \infty c^3)$, and if this be less by unity than the number of the terms of $ce \infty c^3$, say if we have $(1 - D)(ce \infty c^3) = 1$, then there will be a single seminvariant $ce \infty c^3$. We in fact find $(1 - D): (g \infty d^2), (ce \infty c^3), (d^2 \infty b^2c^2), (c^2 \infty b^6)$, each $= 1$ and thus establish the existence of the foregoing seminvariants $g \infty d^2$, $ce \infty c^3$, $d^2 \infty b^2c^2$, $c^3 \infty b^6$. And similarly if in any case we have $(1 - D)(I \infty F) = 2$ or any larger number, then we have 2 or more seminvariants $I \infty F$.

43. It will be convenient to write down at once the system of square diagrams for the several weights 2 to 16; each of these may theoretically be obtained by a direct process of calculation such as I exhibit for the weight 10, but the labor would be very great indeed, and I have in fact formed the squares for the weights 11 to 16, not in this manner but by the MacMahon linkage.

$w = 2$

	b^2
c	1

$w = 3$

	b^3
d	1

$w = 4$

	c^2	b^4
e	1	
c^2		1

$w = 5$

	bc^2	b^5
f	1	
cd		1

$w = 6$

	d^2	c^3	b^2c^2	b^6
g	1			
ce		1		
d^2			1	
c^3				1

$w = 7$

	bd^2	bc^3	b^3c^2	b^7
h	1			
cf		1		
de			1	
c^2d				1

$w = 8$

	e^2	cd^2	b^2d^2	c^4	b^2c^3	b^4c^2	b^8
i	1						
cg		1					
df			1				
e^2				1			
c^2e					1		
cd^2						1	
c^4							1

$w = 9$

	be^2	d^3	bcd^2	b^3d^2	bc^4	b^3c^3	b^5c^2	b^9
j	1							
ch		1						
dg			1					
ef				1				
c^2f					1			
cde						1		
d^3							1	
c^3d								1

$w = 10$

	f^2	ce^2	b^2e^2	bd^3	c^2d^2	b^2cd^2	c^5	b^4d^2	b^2c^4	b^4c^3	b^6c^2	b^{10}
k	1											
ci		1										
dh			1									
eg				1								
c^2g						2						
f^2					1							
cdf								2				
ce^2							1					
d^2e									1			
c^3e										1		
c^2d^2											1	
c^5												1

The subsequent squares $w = 11$ to 16 are for convenience given in the plates at the end of the present memoir.

44. It is to be observed that in each square the outside left-hand terms are the nonunitaries in CO and the outside bottom terms are the power-enders in AO. I have inside each square written down only the significant numbers, but we might fill up the whole square. For instance $w = 7$, the filled-up square would be

h	1	2	3	4
cf	0	1	2	3
de	-1	0	1	2
c^2d	0	0	0	1
	bd^2	bc^3	b^3c^2	b^7

where in the first column the numbers relate to the sets $h \infty bd^2$, $cf \infty bd^2$, $de \infty bd^2$ and $c^2d \infty bd^2$ (this last set $c^2d \infty bd^2$ is non-existent since c^2d is in AO inferior to bd^2, i. e. as well for the set as for the diminished set, number of terms is $= 0$, and we have for the compartment $0 - 0, = 0$). And similarly for the remaining three columns. The process of thus filling up the whole square is a direct and non-tentative one, and the conclusions to which the numbers lead are as follows: col. 1, the final being bd^2, the initial cannot be c^2d, de or cf, but taking it to be h, we have the seminvariant $h \infty bd^2$. Col. 2, the final being bc^3 the initial cannot be c^2d or de, but taking it to be cf we have the seminvariant $cf \infty bc^3$: it may be added that the top number 2 shows that there are two seminvariants $h \infty bc^3$, these are of course the foregoing ones $h \infty bd^2$ and $cf \infty bc^3$. Similarly col. 3, the final being b^3c^2, the initial cannot be c^2d, but taking it to be de, we have the seminvariant $de \infty b^3c^2$, and col. 4, we have the seminvariant $c^2d \infty b^7$.

For the several weights up to 9 we have simply units in the dexter diagonal of each square, viz. the nonunitaries in CO correspond to the power-enders in AO, or the sharp seminvariants are $c \infty b^2$, $d \infty b^3$, etc. See *post*, Table of Reductions, No. 62, which exhibits these correspondences.

45. For the weight 10 we have deviations: the figures 1 and 2 denote as follows:

$1-D$	$k \infty f^2$	$=$	1
	ci " ce^2	"	1
	dh " b^2e^2	"	1
	eg " bd^3	"	1
	f^2 " c^2d^2	"	1
	c^2g " b^2cd^2	"	2
	ce^2 " ce^5	"	1
	cdf " b^4d^2	"	2
	d^2e " b^2c^4	"	1
	c^3e " b^4c^3	"	1
	c^2d^2 " b^6c^2	"	1
	c^5 " b^{10}	"	1

and they indicate the sharp seminvariants $k \infty f^2$, $ci \infty ce^2$, etc.: where observe that the power-enders being in AO as before, the nonunitaries are not in CO, but we have inversions (c^2g, f^2) and (cdf, ce^2).

In particular $(1-D)(f^2 \infty c^2d^2) = 1$ indicates the seminvariant $f^2 \infty c^2d^2$; $(1-D)(c^2g \infty b^2cd^2) = 2$, means in the first instance that there are 2 seminvariants $c^2g \infty b^2cd^2$, but here the set $c^2g \infty b^2cd^2$ includes as part of itself the set $f^2 \infty c^2d^2$; so that if $c^2g \infty b^2cd^2$ is used to denote any particular form, then the general form is $c^2g \infty b^2cd^2$ plus arbitrary multiple of $f^2 \infty c^2d^2$, and we have thus virtually a single form $c^2g \infty b^2cd^2$. And similarly the set $cdf \infty b^4d^2$ includes as part of itself the set $ce^2 \infty c^5$, and thus the general form $cdf \infty b^4d^2$ is $=$ particular form plus arbitrary multiple of $ce^2 \infty c^5$, or we have virtually a single form $cdf \infty b^4d^2$.

I remark that it would be allowable to take as a standard form of $c^2g \infty b^2cd^2$, a form not containing any term in f^2, and similarly for the standard form of $cdf \infty b^4d^2$ a form not containing any term in ce^2; but this is not done in the tables.

46. The diagram for weight 10 is constructed by the following calculation; viz. in col. 1 we calculate $(1-D)$ $(k \infty f^2)$ and for this purpose write down the terms of $k \infty f^2$, and $D(k \infty f^2)$ in CO: in col. 2 we calculate $(1-D)(ci \infty ce^2)$, and for this purpose write down the terms of $k \infty ce^2$ and $D(k \infty ce^2)$ in CO, the terms of $ci \infty ce^2$ and $D(ci \infty ce^2)$ being thence found by rejecting the terms k, bj

and the term j at the head of the two halves of the column. So in col. 3 we calculate $(1-D)(dh \infty b^2e^2)$, and for this purpose write down the terms of $(k \infty b^2e^2)$ and $D(k \infty b^2e^2)$ in CO, and for $dh \infty b^2e^2$ and $D(dh - b^2e^2)$ reject the terms k, bj, ci, b^2i and j, bi at the head of the two halves of the column. And so for the remaining columns. It is to be remarked that there is in each successive column a continually increasing number of terms to be rejected; by a properly devised variation of the algorithm it would have been possible to avoid writing down these terms at all, but for greater clearness I have inserted them.

1		2		3		4		5		6		7		8		9		10		11		12	
k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j	k	j
bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi	bj	bi
ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch	ci	ch
dh	dg	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^2h	b^2i	b^3h	b^2i	b^2h
eg	ef	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg	dh	dg
f^2		bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg	bch	bcg
		eg	ef	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^3h	b^3g	b^2h	b^2g
		bdg	bdf	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef	eg	ef
		c^2g	c^2f	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf	bdg	bdf
		f^2	be^2	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^3f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f	c^2g	c^2f
		bef	cde	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf	b^2cg	b^2cf
		cdf		f^2	be^2	f^2	be^2	f^2	be^2	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f	b^4g	b^4f
		ce^2		bef	cde	bef	cde	bef	cde	f^2	be^2	f^2	be^2	f^2	be^2	f^2	be^2	f^2	be^2	f^2	be^2	f^2	be^2
				cdf	b^2de	cdf	b^2de	cdf	b^2de	bef	cde	bef	cde	bef	cde	bef	cde	bef	cde	bef	cde	bef	cde
				b^2df	d^3	b^2df	bc^2e	b^2df	bc^2e	cdf	b^2de	cdf	b^2de	cdf	b^2de	cdf	b^2de	cdf	b^2de	cdf	b^2de	cdf	b^2de
				ce^2		bc^2f	d^3	bc^2f	d^3	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e	b^2df	bc^2e
				d^2e		ce^2	bcd^2	ce^2	bcd^2	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce	bc^2f	b^3ce
				b^2e^2		b^2e^2		b^2e^2	c^3d	b^3cf	d^3	b^3cf	d^3	b^3cf	b^5e	b^3cf	b^5e	b^3cf	b^5e	b^3cf	b^5e	b^3cf	b^5e
						d^2e		d^2e		ce^2	bcd^2	ce^2	bcd^2	b^5f	d^3	b^5f	d^3	b^5f	d^3	b^5f	d^3	b^5f	d^3
						$bcde$		$bcde$		b^2e^2	b^3d^2	b^2e^2	b^3d^2	ce^2	bcd^2	ce^2	bcd^2	ce^2	bcd^2	ce^2	bcd^2	ce^2	bcd^2
						bd^3		c^3e		d^2e	c^3d	d^2e	c^3d	b^2e^2	b^3d^2	b^2e^2	b^3d^2	b^2e^2	b^3d^2	b^2e^2	b^3d^2	b^2e^2	b^3d^2
								bd^3		$bcde$	b^2c^2d	$bcde$	b^2c^2d	d^2e	c^3d	d^2e	c^3d	d^2e	c^3d	d^2e	c^3d	d^2e	c^3d
								c^2d^2		b^3de		b^3de	bc^4	$bcde$	b^2c^2d	$bcde$	b^2c^2d	$bcde$	b^2c^2d	$bcde$	b^2c^2d	$bcde$	b^2c^2d
										c^3e		c^3e		b^3de	b^4cd	b^3de	b^4cd	b^3de	b^4cd	b^3de	b^4cd	b^3de	b^4cd
										b^2c^2e		b^2c^2e		c^3e	bc^4	c^3e	bc^4	c^3e	b^6d	c^3e	b^6d	c^3e	b^6d
										bd^3		bd^3		b^2c^2e		b^2c^2e	b^3c^3	b^2c^2e	bc^4	b^2c^2e	bc^4	b^2c^2e	bc^4
										c^2d^2		c^2d^2		b^4ce		b^4ce		b^4ce	b^3c^3	b^4ce	b^3c^3	b^4ce	b^3c^3
										b^2cd^2		b^2cd^2		bd^3		bd^3		b^6e	b^5c^2	b^6e	b^5c^2	b^6e	b^5c^2
												bc^3d		c^2d^2		c^2d^2		bd^3		bd^3	b^7c	bd^3	b^7c
												c^5		b^2cd^2		b^2cd^2		c^2d^2		c^2d^2		c^2d^2	b^9
														b^4d^2		b^4d^2		b^2cd^2		b^2cd^2		b^2cd^2	
														bc^3d		bc^3d		b^4d^2		b^4d^2		b^4d^2	
														c^5		b^3c^2d		bc^3d		bc^3d		bc^3d	
																c^5		b^3c^2d		b^3c^2d		b^3c^2d	
																b^2c^4		b^5cd		b^5cd		b^5cd	
																		c^5		b^7d		b^7d	
																		b^2c^4		c^5		c^5	
																		b^4c^3		b^2c^4		b^2c^4	
																				b^4c^3		b^4c^3	
																				b^6c^2		b^6c^2	
																						b^8c	
																						b^6	
$6-5=$		$11-10=$		$14-13=$		$14-13=$		$12-11=$		$19-17=$		$12-11=$		$19-17=$		$14-13=$		$14-13=$		$11-10=$		$6-5=$	
1		1		1		1		1		2		1		2		1		1		1		1	

47. As to the first of the foregoing inversions c^2g, f^2, it is proper to remark, that filling up two compartments of the square we have

c^2g		1	2
f^2		1	1
		c^2d^2	b^2cd^2

where the meaning of the numbers (1, 1) has to be considered: the first (1) seems to indicate a seminvariant $c^2g \propto c^2d^2$, but there is in fact no such form, what it really indicates is a form $0c^2g + f^2 \propto c^2d^2$, that is $f^2 \propto c^2d^2$; and similarly the second (1) seems to indicate a seminvariant $f^2 \propto b^2cd^2$, but there is in fact no such form, what it really indicates is $f^2 \propto c^2d^2 + 0b^2cd^2$, that is $f^2 \propto c^2d^2$. The explanation is correct, but to make it perfectly clear some further developments would be required. The like remarks apply to the inversion cdf, ce^2.

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48. We require the two theorems:

The first is: if a seminvariant S has q for its highest letter, then $\partial_q S$ is also a seminvariant.

The second has presented itself for unitariants (*ante* No. 31); for seminvariants the form is less simple, viz. If in any seminvariant, attending only to the terms of the highest degree, we therein change $b, c, d, e, \ldots$ into $b, 2c, 6d, 24e, \ldots$ and then diminish the letters (that is replace each letter by the next preceding letter) and in the result so obtained change $b, c, d, e, \ldots$ into $b, \frac{c}{2}, \frac{d}{6}, \frac{e}{24}, \ldots$ we obtain a seminvariant. For instance $g - 6bf + 15ce - 10d^2$, in the terms of degree 2, making the numerical change we have $-720bf + 720ce - 360d^2$, and then diminishing the letters and making the numerical change, we obtain $-720\frac{e}{24} + 720\frac{bd}{6} - 360\frac{c^2}{4}$, that is $-30(e - 4bd + 3c^2)$, a seminvariant.

For the proof observe that the equation $\Delta S = 0$, attending therein only to the terms of the highest degree gives $(2b\partial_c + 3c\partial_d + \ldots)S' = 0$, if S' denote the terms of the highest degree: making the numerical change, this is

$(b\partial_c + c\partial_d + \ldots .)S''$, if S'' is what S' becomes thereby; diminishing the letters this is $(\partial_b + b\partial_c + \ldots .)S''' = 0$, if S''' is the diminished value of S'', and finally making the numerical change, if T be what S''' becomes on writing therein $b, \frac{c}{2}, \frac{d}{6}, \ldots .$ for $b, c, d, \ldots .$ this gives $(\partial_b + 2b\partial_c + \ldots .)\, T = 0$, viz. T is a seminvariant.

49. Assume that for the weights up to a certain weight w the forms of the sharp seminvariants are known: and for the weight w consider a seminvariant $I(\text{ca})F$: here if I be given, the first theorem establishes a limit F' such that F is in AO not higher than F'. For instance $w = 12$, if $I = dj$, the coefficient of j as being a seminvariant can only be $d \infty b^3$, and thus the seminvariant contains a term $b^3 j$, or the final term F must be in AO not higher than $b^3 j$; the degree is thus $= 4$ at least.

Similarly if F be given, then the second theorem determines a limit I' such that I is in CO not lower than I'. Thus $w = 12$, as before, if $F = b^4 cd^2$, then diminishing the letters we have bc^2, a term belonging to $f \infty bc^2$; the diminished form has thus terms $a^4(a^2 f,\ bc^2)$, so that augmenting these the seminvariant has terms $b^4(b^2 g,\ cd^2)$ and thus the initial term I is in CO not lower than $b^6 g$.

50. A limit for I or F when the other is given can also in some cases be found as follows: Considering a seminvariant of the weight w as before, and denoting its extent and degree by σ and δ respectively, then we have $\sigma\delta - 2w = 0$ or positive; that is $\sigma\delta = 2w$ at least; here given I, we have σ, and then $\delta = \frac{2w}{\sigma}$ at least; and given F we have δ, and then $\sigma = \frac{2w}{\delta}$ at least.

51. We may now explain the MacMahon linkage; for a given weight we write down in two columns the initials or nonunitaries in CO, and the finals or power-enders in AO: by what precedes it appears that we cannot combine the terms of the one column each with the term opposite to it in the other column; what we do is: beginning with the top of the column of initials we combine successively each term with the highest admissible term in the column of finals: or beginning with the bottom of the column of finals, we combine successively each term with the lowest admissible term in the column of initials.

52. For the weight 12, the linkage is

read downwards.

shown by	not in AO higher than				
		m	g^2	bl	$k \infty f^2$
$(c \infty b^2)k$	b^2k	ck	cf^2	b^2k	j " be^2
$(d$ " $b^3)j$	b^3j	dj	e^3	bdi	ch " d^3
$(e$ " $c^2)i$	c^2i	ei	b^2f^2	b^3j	i " e^2
$(c^2$ " $b^4)i$	b^4i	c^2i	bde^2	b^2dh	cg " cd^2
$(f$ " $bc^2)h$	bc^2h	fh	c^2e^2	b^2eg	df " b^2d^2
$(cd$ " $b^5)h$	b^5h	cdh	d^4	b^2f^2	e^2 " c^4
$(g$ " $d^2)g$	d^2g	g^2	b^2ce^2	b^4i	h " bd^2
$(ce$ " $c^3)g$	c^3g	ceg	bcd^3	b^3dg	cf " bc^3
$(d^2$ " $b^2c^2)g$	b^2c^2g	d^2g	c^3d^2	b^3ef	de " b^3
$(c^3$ " $b^6)g$	b^6g	c^3g	b^4e^2	b^5h	g " d^2
$(cf$ " $bc^3)f$	bc^3f	cf^2	b^3d^3	b^4df	ce " c^3
$(de$ " $b^3c^2)f$	b^3c^2f	def	$b^2c^2d^2$	b^4e^2	d^2 " b^2c^2
$(c^2d$ " $b^7)f$	b^7f	c^2df	c^6	b^3d^3	c^3 " b^6
$(e^2$ " $c^4)e$	c^4e	e^3	b^4cd^2	b^6g	f " bc^2
$(c^2e$ " $b^2c^3)e$	b^2c^3e	c^2e^2	b^2c^5	b^5de	cd " b^5
$(cd^2$ " $b^4c^2)e$	b^4c^2e	cd^2e	b^6d^2	b^7f	e " c^2
$(c^4$ " $b^8)e$	b^8e	c^4e	b^4c^4	b^6d^2	c^2 " b^4
$(d^3$ " $b^5c^2)d$	b^5c^2d	d^4	b^6c^3	b^8e	d " b^3
$(c^2d$ " $b^9)d$	b^9d	c^3d^2	b^8c^2	b^9d	c " b^2
$(c^5$ " $b^{10})c$	$b^{10}c$	c^6	b^{12}		
				lower than not in CO	shown by

read upwards.

Thus, beginning at the top of the column of initials, m is to be linked with g^2, that is we have $(m \infty g^2)$; ck with cf^2, that is we have $(ck \infty cf^2)$; dj cannot be linked with i^3, for the final must be in AO not higher than b^3j, but it is linked with the highest term b^2f^2 for which this condition is satisfied, that is we have $(dj \infty b^2f^2)$; ei is then linked with the highest admissible term e^3, that is we have $(ei \infty e^3)$; and so on.

Or beginning at the bottom of the column of finals b^{12} is linked with c^6, that is we have $(c^6 \infty b^{12})$, b^8c^2 with c^3d^2, that is we have $(c^3d^2 \infty b^8c^2)$; b^6c^3 cannot be linked with d^4, for the initial must be in CO not lower than b^8e, but it is linked with the lowest term c^4e for which this condition is satisfied, that is we have $(c^4e \infty b^6c^3)$; and so on.

The Umbral Notation. Stroh's Theory. Art. Nos. 53 *to* 56.

53. Employing the umbræ α, β, γ, δ, ... which are such that $\alpha = \beta = \gamma, \ldots\ldots = b$; $\alpha^2 = \beta^2 = \gamma^2, \ldots\ldots = c$; $\alpha^3 = \beta^3 = \gamma^3, \ldots\ldots = d$; and so on, then for instance $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2, = c - 2b^2 + c, = 2(c - b^2)$, a seminvariant; $(\alpha - \beta)^2(\alpha - \gamma) = \alpha^3 - 2\alpha^2\beta + \alpha\beta^2 - \alpha^2\gamma + 2\alpha\beta\gamma - \beta^2\gamma, = d - 2bc + bc - bc + 2b^3 - bc, = d - 3bc + 2b^3$, a seminvariant: and so in general any rational and integral function of the differences of the umbræ developed and interpreted is a seminvariant. For the seminvariants of a given weight e. g. $w = 6$, Dr. Stroh* considers the function $\Omega^6 = (\alpha x + \beta y + \gamma z + \delta w + \epsilon t + \zeta u)^6$ where x, y, z, w, t, u are numbers the sum of which is $= 0$, or we may if we please have more than 6 such numbers: the expression is obviously a function of the differences of the umbræ and it is thus a seminvariant. To develop its value observe that after expansion of the sixth power we have sets of similar terms, for instance $\alpha^6x^6 + \beta^6y^6 + \ldots\ldots$ which putting therein $\alpha^6 = \beta^6 = \gamma^6, \ldots\ldots = g$ become $= g \,.\, Sx^6$, and generally each set becomes equal to a literal term multiplied by a symmetric function of the x, y, z, w, ...; introducing capital letters to denote the elementary symmetric functions of these quantities, and recollecting that their sum is assumed to be $= 0$, say we have

$$1 + C\vartheta^2 + D\vartheta^3 + E\vartheta^4 + \ldots\ldots = 1 - x\vartheta \,.\, 1 - y\vartheta \,.\, 1 - z\vartheta \ldots\ldots$$

(that is $0 = Sx$, $+C = Sxy$, $-D = Sxyz$, etc.) then by aid of the Table VI(b) writing therein 0, C, D, E, F, G for b, c, d, e, f, g we find

* See the paper "Ueber die Symbolische Darstellung der Grundsyzyganten einer binären Form sechster Ordnung und eine Erweiterung der Symbolik von Clebsch," *Math. Ann.* t. XXXVI, 1890, pp. 263-303, in particular §10, Das Formensystem einer Form unbegrenzt hoher Ordnung.

$\Omega^6 = (\alpha x + \beta y + \gamma z \ldots .)^6 = \alpha^6 Sx^6 + 6\alpha^5\beta Sx^5y +$ etc., as shown in the following table:

			C^3	D^2	CE	G
$1\,g$	Sx^6	=	-2	$+3$	$+6$	-6
$+\ 6\,bf$	Sx^5y	=	$+2$	-3	-6	$+6$
$+\ 15\,ce$	Sx^4y^2	=	-2	-3	$+2$	$+6$
$+\ 20\,d^2$	Sx^3y^3	=	$+1$	$+3$	-3	$+3$
$+\ 30\,b^2e$	Sx^4yz	=		$+3$	$+2$	-6
$+\ 60\,bcd$	Sx^3y^2z	=		-3	$+4$	-12
$+\ 90\,c^3$	$Sx^2y^2z^2$	=		$+1$	-2	-2
$+120\,b^3d$	Sx^3yzw	=			-2	$+6$
$+180\,b^2c^2$	Sx^2y^2zw	=			$+1$	$+9$
$+360\,b^4c$	Sx^2yzwt	=				-6
$+720\,b^6$	$Sxyzwtu$	=				$+1$
			$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

the numbers whereof are it will be observed identical with those of the foregoing table No. 33, relating to the MacMahon equation.

This is to be read

$$\Omega^6 = C^3[d^2] + D^2[c^3] + CE[b^2c^2] + G[b^6]$$

viz. Ω^6 is a linear function of C^3, D^2, CE and G, the coefficients of these, being given functions of (b, c, d, e, f, g), which given functions are the specific blunt seminvariants which have been already called $[d^2]$, $[c^3]$, $[b^2c^2]$ and $[b^6]$. And so in general the developed value of Ω^w affords a complete definition of these specific blunt seminvariants of the weight w. Observe that α, β, γ, δ, ... are umbræ in nowise connected with the roots α, β, γ, δ, ... before made use of, and that B, C, D, ... are actual quantities in nowise connected with the symbolic capitals B, C, D, ... before made use of.

54. The capital and small letter symbols are conjugate to each other. It will be convenient to give here, in reference to subsequent investigations a table of these conjugate forms up to the degree 6 and weight 15.

Table of Conjugates.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
C b^2	D b^3	C^2 c^2	CD bc^2	C^3 d^2	C^2D bd^2	C^4 e^2	C^3D be^2	C^5 f^2	C^4D bf^2	C^6 g^2	C^5D bg^2	C^7 h^2	C^6D bh^2
		E b^4	F b^5	D^2 c^3	DE bc^3	CD^2 cd^2	D^3 d^3	C^2D^2 ce^2	CD^3 de^2	C^3D^2 cf^2	C^2D^3 df^2	C^4D^2 cg^2	C^3D^3 dg^2
				CE b^2c^2	CF b^3c^2	C^2E b^2d^2	CDE bcd^2	C^3E b^2e^2	C^2DE bce^2	D^4 e^3	C^3DE bcf^2	CD^4 ef^2	D^5 f^3
				G b^6		E^2 c^4	C^2F b^3d^2	D^2E bd^3	DE^2 cd^3	C^4E b^2f^2	D^3E be^3	C^5E b^2g^2	C^4DE bcg^2
						DF b^2c^3	EF bc^4	CE^2 c^2d^2	C^2F b^3e^2	CD^2E bde^2	CDE^2 cde^2	C^2D^2E bdf^2	CD^3E bef^2
						CG b^4c^2	DG b^3c^3	CDF b^2cd^2	D^2F b^2d^3	C^2E^2 c^2e^2	C^4F b^3f^2	C^3E^2 c^2f^2	C^2DE^2 cdf^2
								F^2 c^5	CEF bc^2d^2	E^3 d^4	CD^2F b^2de^2	D^2E^2 ce^3	DE^3 de^3
								C^2G b^4d^2	CDG b^3cd^2	C^2DF b^2ce^2	C^2EF bc^2e^2	CE^3 d^2e^2	C^5F b^3g^2
								DG b^2c^4	FG bc^5	DEF bcd^3	E^2F bd^4	C^3DF b^2cf^2	C^2D^2F b^2df^2
										CF^2 c^3d^2	DF^2 c^2d^3	D^3F b^2e^3	C^3EF bc^2f^2
										C^3G b^4e^2	C^2DG b^3ce^2	$CDEF$ $bcde^2$	D^2EF bce^3
										D^2G b^3d^3	DEG b^2cd^3	C^2F^2 c^3e^2	CE^2F bd^2e^2
										CEG $b^2c^2d^2$	CFG bc^3d^2	EF^2 cd^4	CDF^2 c^2de^2
										G^2 c^6		C^4G b^4f^2	F^3 d^5
												CD^2G b^3de^2	C^3DG b^3cf^2
												C^2EG $b^2c^2e^2$	D^3G b^3e^3
												E^2G b^2d^4	$CDEG$ b^2cde^2
												DFG bc^2d^3	C^2FG bc^3e^2
												CG^2 c^4d^2	EFG bcd^4
													DG^2 c^3d^3

55. We can by means of the umbral notation write down for the blunt seminvariants of a given weight (indefinite forms, not the above mentioned specific forms) expressions far more simple than those which are given by the foregoing theories: we can in fact find without difficulty *monomial* umbral expressions; and in many cases obtain also the sharp forms. To illustrate this, I consider the weight 10: I write down for convenience the symbols of the sharp forms (though the knowledge of these is in nowise required) and I form a table as follows:

Sharp forms, finals in AO.		
$k \infty f^2$	1	$(\alpha-\beta)^{10}$
ci " ce^2	2	$(\alpha-\beta)^8(\alpha-\gamma)^2$
dh " b^2e^2	3	$(\alpha-\beta)^8(\alpha-\gamma)(\alpha-\delta)$
eg " bd^3	4	$(\alpha-\beta)^6(\alpha-\gamma)^3(\alpha-\delta)$
f^2 " c^2d^2	5	$(\alpha-\beta)^6(\alpha-\gamma)^2(\alpha-\delta)^2$
c^2g " b^2cd^2	6	$(\alpha-\beta)^6(\alpha-\gamma)^2(\alpha-\delta)(\alpha-\varepsilon)$
ce^2 " c^5	7	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)^2(\alpha-\varepsilon)^2$
cdf " b^4d^2	8	$(\alpha-\beta)^6(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta)$
d^2e " b^2c^4	9	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)^2(\alpha-\varepsilon)(\alpha-\zeta)$
c^3e " b^4c^3	10	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta)(\alpha-\eta)$
c^3d^2 " b^6c^2	11	$(\alpha-\beta)^4(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta)(\alpha-\eta)(\alpha-\theta)$
c^5 " b^{10}	12	$(\alpha-\beta)^2(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)(\alpha-\zeta)(\alpha-\eta)(\alpha-\theta)(\alpha-\iota)(\alpha-\kappa)$

It will be observed that all the differences used are $\alpha-\beta$, $\alpha-\gamma$, ... containing each of them an α; hence in all the forms we have α^{10}, $=k$; in $(\alpha-\beta)^{10}$ the lowest term (in AO) is $\alpha^5\beta^5$, $=f^2$; in $(\alpha-\beta)^8(\alpha-\gamma)^2$, the lowest term is $\alpha^4\beta^4 . \gamma^2$, $=ce^2$; and so on, viz. in each case the lowest term is the final term of the sharp form set down in the same line.

56. The form $(\alpha-\beta)^{10}$ gives at once the sharp form $k \infty f^2$; we thus develop it:

α^{10}	$\alpha^9\beta$	$\alpha^8\beta^2$	$\alpha^7\beta^3$	$\alpha^6\beta^4$	$\alpha^5\beta^5$
β^{10}	$\alpha\beta^9$	$\alpha^2\beta^8$	$\alpha^3\beta^7$	$\alpha^4\beta^6$	
1	-10	$+45$	-120	$+210$	-252
$+1$	-10	$+45$	-120	$+210$	
$=2(k$	$-10bj$	$+45ci$	$-120dh$	$+210eg$	$-126f^2)$

$(\alpha-\beta)^8(\alpha-\gamma)^2$ contains a term α^{10}, $=k$ and thus gives a blunt form $k\alpha o c e^2$; if instead of it we employ the form $(\alpha-\beta)^8(\alpha-\gamma)(\beta-\gamma)$, then here as before the lowest term is $\alpha^4\beta^4\,.\,\gamma^2$, $=ce^2$, but there is no term α^{10}: there is a term $\alpha^9\beta$, $=bj$, but as this cannot appear, we must have terms of this form destroying each other. The simplest mode of effecting the development is to write $(\alpha-\beta)^8(\alpha-\gamma)(\beta-\gamma)=(\alpha-\beta)^8\{\alpha\beta-\gamma(\alpha+\beta)+\gamma^2\}$, we may herein put at once $\gamma=b$, $\gamma^2=c$, and thus the form is $(\alpha-\beta)^8\{\alpha\beta-b(\alpha+\beta)+c\}$; I develop thus:

$$
\begin{array}{ll}
(\alpha-\beta)^8 & 1,\ -8,\ +28,\ -56,\ +70,\ -56,\ +28,\ -8,\ +1, \\
 & \quad +1,\ -8,\ +28,\ -56,\ +70,\ -56,\ +28,\ -8,\ +1 \\
\hline
(\alpha-\beta)^8(\alpha+\beta) & 1,\ -7,\ +20,\ -28,\ +14,\ +14,\ -28,\ +20,\ -7,\ +1
\end{array}
$$

$$
\begin{array}{l}
\quad 1bj - 8ci + 28dh - 56eg + 70f^2 \\
\quad +1 \quad -8 \quad +28 \quad -56 \\
-b\left(\begin{array}{l} 1j - 7bi + 20ch - 28dg + 14ef \\ +1 \quad -7 \quad +20 \quad -28 \quad +14 \end{array}\right) \\
+c\left(\begin{array}{l} 1i - 8bh + 28cg - 56df + 70e^2 \\ +1 \quad -8 \quad +28 \quad -56 \end{array}\right)
\end{array}
$$

					$\div -14$
k					
bj	$+2$	-2		0	
ci	-16		$+2$	-14	$+1$
dh	$+56$			$+56$	-4
eg	-112			-112	$+8$
f^2	$+70$			$+70$	-5
b^2i		$+14$		$+14$	-1
bch		-40	-16	-56	$+4$
bdg		$+56$		$+56$	-4
bef		-28		-28	$+2$
c^2g			$+56$	$+56$	-4
cdf			-112	-112	$+8$
ce^2			$+70$	$+70$	-5
					± 23

which in fact exhibits the calculation of the sharp form $ci \infty ce^2$. The disappearance of the term in bj will be noticed.

Instead of $(\alpha-\beta)^8(\alpha-\gamma)(\beta-\delta)$ which contains α^{10} that is k, we may take $(\alpha-\beta)^8(\gamma-\delta)^2$ that is $(i-8bh+28cg-56df+35e^2)(c-b^2)$: this is $ci\alpha o b^2e^2$, a blunt form; by subtracting from it $ci \infty ce^2$, we could obtain the next sharp form $dh \infty b^2e^2$; but this in passing; it does not appear that there is any monomial umbral expression for the last-mentioned form.

I do not stop to examine the next following forms, but pass on at once to the last of them; instead of the expression given we may take the expression $(\alpha-\beta)^2(\gamma-\delta)^2(\varepsilon-\zeta)^2(\eta-\theta)^2(\iota-\varkappa)^2$, that is $(c-b^2)^5$, which is in fact the sharp form $c^5 \infty b^{10}$.

Seminvariants of a given Degree: Generating Functions. Art. Nos. 57 *to* 59.

57. We may consider the seminvariants of a given degree, and arrange them according to their weights: thus in each case writing down the series of finals, and for a reason that will appear also the conjugates of these finals (see Table of Conjugates, *ante* No. 54).

For degree 2, or quadric seminvariants, we have

2	3	4	5	6 ....
$C,\ b^2$	—	$C^2,\ c^2$	—	$C^3,\ d^2$

there is here for every even weight (beginning with 2) a single form, and for every odd weight no form: the number of forms of the weight w is thus $=$ coeff. of x^w in $x^2 \div (1-x^2)$, or writing for shortness $\mathbf{2}$ to denote $1-x^2$, (and similarly $\mathbf{3}$, $\mathbf{4}$, ... to denote $1-x^3$, $1-x^4$, ...) say that for degree 2, Generating Function, *G. F.*, is $= x^2 \div \mathbf{2}$.

For degree 3, or cubic seminvariants, we have

3	4	5	6	7 ....
$D,\ b^3$	—	$CD,\ bc^2$	$D^2,\ c^3$	$C^2D,\ bc^3$

the counting is most easily effected by means of the conjugate forms; these contain all of them the factor D, and omitting this factor we have all the combinations of C, D which make up the weight $w-3$, viz. for weight w, we have number of ways in which $w-3$ can be made up with the parts 2, 3: that is,

$$\text{for degree } 3,\ G.\ F. \text{ is } = x^3 \div \mathbf{2}\cdot\mathbf{3}.$$

Similarly for degree 4 or quartic seminvariants, we have terms each containing E, and removing this factor, we have all the combinations of C, D, E which make up the weight $w-4$, viz.

$$\text{for degree } 4,\ G.\ F. \text{ is } = x^4 \div \mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}.$$

Thus for degrees

2, 3, 4, 5, 6, ...

the *G. F.*'s are

$$= x^2 \div \mathbf{2},\ x^3 \div \mathbf{2}\cdot\mathbf{3},\ x^4 \div \mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4},\ x^5 \div \mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{5},\ x^6 \div \mathbf{2}\cdot\mathbf{3}\cdot\mathbf{4}\cdot\mathbf{5}\cdot\mathbf{6},\ \ldots.$$

58. We may analyse these results by separating the finals into classes. I use the expression $b, c, d, \ldots.$ are discrete letters, meaning thereby that they are distinct letters, not of necessity consecutive but with any intervals between them. Thus deg. 3, if (b, c) are discrete letters, then the finals are b^3, and bc^2; deg. 4, if b, c, d are discrete letters then the finals are b^4, bc^3, b^2c^2, and bcd^2; and so on, the number of classes being doubled at each step, as will presently appear for the weights 5 and 6 respectively.

I notice also a property of the conjugates of these classes; for b^3 and bc^2 themselves the conjugates are D, and CD, and these occur as factors, D in the conjugate of every form of the class b^3 (for instance conjugates of c^3, d^3 are D^2, D^3) and CD in the conjugate of every form of the class bc^2 (for instance conjugates of bd^2, ce^2 are C^2D, C^2D^2); and the like in other cases, viz. for any class whatever the conjugate of the first or representative form occurs as a factor in the conjugates of the several other forms belonging to the same class.

59. With these explanations, the expressions for the several $G.\,F.$'s are obtained without difficulty, and we have

$$\begin{array}{llll} \text{deg. } 2, & \text{class} & C,\ b^2 & G.F. = x^2 \div 2 \\ \text{deg. } 3, & \text{class} & D,\ b^3 & G.F. = x^3 \div 3 \\ & \text{"} & CD,\ bc^2 & \text{"}\quad \text{"}\ x^5 \div 2\cdot 3 \end{array}$$

we ought here to have

$$x^3 \div 2\cdot 3 = x^3 \div 3 + x^5 \div 2\cdot 3 \text{ viz. in verification}$$

$$\begin{array}{lll} x^3 & = x^3 . 2 & = x^3 - x^5 \\ & \ \ + x^5 & \ \ \ \ \ \ \ + x^5 \\ \hline & & = x^3 \end{array}$$

$$\begin{array}{llll} \text{deg. } 4, & \text{class} & E,\ b^4 & G.F. = x^4 \div 4 \\ & \text{"} & DE,\ bc^3 & \text{"}\quad \text{"}\ x^7 \div 3\cdot 4 \\ & \text{"} & CE,\ b^2c^2 & \text{"}\quad \text{"}\ x^6 \div 2\cdot 4 \\ & \text{"} & CDE,\ bcd^2 & \text{"}\quad \text{"}\ x^9 \div 2\cdot 3\cdot 4 \end{array}$$

we ought here to have

$$x^4 \div 2\cdot 3\cdot 4 = x^4 \div 4 + x^7 \div 3\cdot 4 + x^6 \div 2\cdot 4 + x^9 \div 2\cdot 3\cdot 4 \text{ viz. in verification}$$

$$\begin{array}{lll} x^4 & = x^4 . 2\cdot 3 & = x^4 - x^6 - x^7 + x^9 \\ & \ \ + x^7 . 2 & \qquad\qquad + x^7 - x^9 \\ & \ \ + x^6 . 3 & \qquad + x^6 \qquad - x^9 \\ & \ \ + x^9 & \qquad\qquad\qquad + x^9 \\ \hline & & = x^4 \end{array}$$

deg. 5, class	$F,$	b^5	$G.F. = x^5 \div 5$
	$EF,$	bc^4	$x^9 \div 4 \cdot 5$
	$DF,$	b^2c^3	$x^8 \div 3 \cdot 5$
	$CF,$	b^3c^2	$x^7 \div 2 \cdot 5$
	$DEF,$	bcd^3	$x^{12} \div 3 \cdot 4 \cdot 5$
	$CEF,$	bc^2d^2	$x^{11} \div 2 \cdot 4 \cdot 5$
	$CDF,$	b^2cd^2	$x^{10} \div 2 \cdot 3 \cdot 4$
	$CDEF,$	$bcde^2$	$x^{14} \div 2 \cdot 3 \cdot 4 \cdot 5$

and for the sum of the eight terms

$G.F. = x^5 \div 2 \cdot 3 \cdot 4 \cdot 5$, which may be verified as before.

deg. 6, class	$G,$	b^6	$G.F. = x^6 \div 6$
	$FG,$	bc^5	$x^{11} \div 5 \cdot 6$
	$EG,$	b^2c^4	$x^{10} \div 4 \cdot 6$
	$DG,$	b^3c^3	$x^9 \div 3 \cdot 6$
	$CG,$	b^4c^2	$x^8 \div 2 \cdot 6$
	$EFG,$	bcd^4	$x^{15} \div 4 \cdot 5 \cdot 6$
	$DFG,$	bc^2d^3	$x^{14} \div 3 \cdot 5 \cdot 6$
	$CFG,$	bc^3d^2	$x^{13} \div 2 \cdot 5 \cdot 6$
	$DEG,$	b^2cd^3	$x^{13} \div 3 \cdot 4 \cdot 6$
	$CEG,$	$b^2c^2d^2$	$x^{12} \div 2 \cdot 4 \cdot 6$
	$CDG,$	b^3cd^2	$x^{11} \div 2 \cdot 3 \cdot 6$
	$DEFG,$	$bcde^3$	$x^{18} \div 3 \cdot 4 \cdot 5 \cdot 6$
	$CEFG,$	bcd^2e^2	$x^{17} \div 2 \cdot 4 \cdot 5 \cdot 6$
	$CDFG,$	bc^2de^2	$x^{16} \div 2 \cdot 3 \cdot 5 \cdot 6$
	$CDEG,$	b^2cde^2	$x^{15} \div 2 \cdot 3 \cdot 4 \cdot 6$
	$CDEFG,$	$bcdef^2$	$x^{20} \div 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$

and for the sum of the sixteen terms

$G.F. = x^6 \div 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$, which may be verified as before.

Reducible Seminvariants—Perpetuants. Art. Nos. 60 *to* 64.

60. Seminvariants of the degrees 2 and 3 are irreducible—or say they are perpetuants. Hence by what precedes, as regards perpetuants

for degree 2, $G.F. = x^2 \div 2$; for degree 3, $G.F. = x^3 \div 2 \cdot 3$.

For the degree 4 (if as before b, c, d denote discrete letters) then the finals are b^4, bc^3, b^2c^2 and bcd^2. For a final b^4, $= b^2 . b^2$ or $b^2c^2 = b^2 . c^2$ we have evidently a product of two quadric seminvariants ending in b^2 and b^2, or in b^2 and c^2, with the

same final term as the quartic seminvariant; so that considering the quartic seminvariants arranged with their finals in AO, and adding to such quartic seminvariant a proper numerical multiple of the product in question, we obtain a quartic seminvariant the final term whereof is in AO higher than the original final term b^4 or b^2c^2, and such quartic seminvariant is thus said to be reducible; a quartic seminvariant not thus reducible is a perpetuant. The quartic perpetuants are consequently those which end in bc^3 or bcd^2. The lowest form is that ending in bc^3, of the weight 7. Taking the sum of the $G. F.$'s for the forms bc^3 and bcd^2 respectively, the $G. F.$ for quartic perpetuants is

$$x^7 \div 3 \cdot 4 + x^9 \div 2 \cdot 3 \cdot 4, \text{ viz. this is } x^7(1 - x^2) + x^9 \div 2 \cdot 3 \cdot 4 \text{ or finally}$$
$$G. F. = x^7 \div 2 \cdot 3 \cdot 4.$$

As an instance of a reduction we have $(d^2 \infty b^2c^2) - (c \infty b^2)(e \infty c^2) = (ce \infty c^3)$, viz. this is $(d \infty b^2c^2) = (c - b^2)(e - 4bd + 3c^2) - (ce - d^2 - b^2e + 2bcd - c^3)$. We have also $(d^2 \infty b^2c^2) = (d \infty b^3)^2 + 4(c \infty b^2)^3$, viz. $(d \infty b^2c^2) = (d - 3bc + 2b^3)^2 + 4(c - b^2)^3$, but this is *not* a reduction, there are on the right-hand side terms of the degree 6, which is higher than the degree of the seminvariant $d^2 \infty b^2c^2$. In general we say that a seminvariant of any given degree is reducible when we can by adding to it products *of its own degree* of seminvariants of inferior degrees reduce it to a seminvariant the final of which is in AO higher than the original final.

61. For the degree 5 (taking b, c, d, e to denote discrete letters) if the final be b^5, bc^4, b^2c^3, b^3c^2, bc^2d^2 or b^2cd^2, then the seminvariant will be reducible; a perpetuant must have therefore a final bcd^3 or $bcde^2$. But it is not true that every quintic seminvariant with either of these finals is a perpetuant. To explain this observe that the first mentioned six finals are some of them in one way only, some of them in two ways, expressible as a product of power-enders, or say they are singly, or else doubly, composite: viz. we have $b^5 = b^2 . b^3$; $bc^4 = c^2 . bc^2$; $b^2c^3 = b^2 . c^3$; $b^3c^2 = c^2 . b^3 = b^2 . bc^2$; $bc^2d^2 = c^2 . bd^2 = d^2 . bc^2$; $b^2cd^2 = b^2 . cd^2$. For a doubly composite form for instance b^3c^2, forming first the product of the quadric and cubic seminvariants ending in c^2, b^3 respectively, and secondly the product of the quadric and cubic seminvariants ending in b^2 and bc^2 respectively, we have two products each with the final b^3c^2, and forming a linear combination so as to eliminate this term b^3c^2, we have thus it may be a quintic seminvariant with a final such as bcd^3 or $bcde^2$, and the process then furnishes a reduction of such a quintic seminvariant. Or on the other hand it may be that the finals of the degree 5 all of them disappear, and we have a relation between products of the form in question (i. e. of a quadric and a cubic seminvariant) and seminvariants of a degree inferior to 5, say this is a quintic syzygy.

In particular a noncomposite final first presents itself for the weight 12, viz. here the finals are b^2ce^2, bcd^3, c^3d^2, the last of these is doubly composite, and it furnishes a reduction of bcd^3. For the weight 13, the finals are b^3f^2, b^2de^2, bc^2e^2, bd^4, c^2d^3 which are each of them singly or doubly composite: for the weight 14 they are b^2cf^2, b^2c^3, $bcde^2$, c^3e^2 and cd^4, and here the doubly composite form furnishes a reduction of $bcde^2$. For the weight 15 we have a final bce^3 which gives a quintic perpetuant. I have in fact in my paper "A Memoir on Seminvariants," Amer. Math. Jour. t. VII (1885), pp. 1–25, worked out the theory of quintic syzygies and perpetuants, and subsequently connecting this with the present theory of finals, I succeeded in showing that when the doubly composite final contains a b then there is not a reduction but a syzygy; we thus have

$$G.F. \text{ for finals } b^3c^2,\ b^3d^2,\ \ldots\ldots = x^7 \div 2$$

$$\text{" \quad " } bc^2d^2,\quad \ldots\ldots = x^{11} \div 2\cdot 4$$

whence for the two forms

$$G.F. \text{ is } x^7 \div 2 + x^{11} \div 2\cdot 4 = \{x^7(1-x^4)+x^{11}\} \div 2\cdot 4,$$

or say for S_5, the number of quintic syzygies $G.F.$ is $= x^7 \div 2\cdot 4$.

I further satisfied myself that the finals for the quintic perpetuants are $bc0e^3$, and $bc0ef^2$, viz. the b, c, e, f being discrete letters, the interposed 0 denotes that the c and e are not consecutive letters. The conjugates of these forms contain the factors D^2EF and CD^2EF respectively and it hence appears that the $G.F.$'s are $= x^{15} \div 3\cdot 4\cdot 5$ and $x^{17} \div 2\cdot 3\cdot 4\cdot 5$; adding these we find

$$\text{for quintic perpetuants } G.F. \text{ is } = x^{15} \div 2\cdot 3\cdot 4\cdot 5,$$

which expression was given in the memoir just referred to: the result was obtained by investigating in the first instance an expression for S_5, the number of quintic syzygies of a given weight. The course of Stroh's investigation to be presently given is different; he determines directly the number of perpetuants, and we may if we please use conversely this result to obtain the number of syzygies.

62. The foregoing theory of reduction is independent of the form of the seminvariants, which may be blunt or sharp at pleasure: the actual formulæ will of course be different, and they are very much more simple for the sharp seminvariants, viz. here in many cases a seminvariant is found to be equal to a product of seminvariants of inferior degrees. I subjoin the following table of the reduction of the several sharp seminvariants up to the weight 12; the forms referred to are the tabulated forms, and to mark that this is so I write down in each case the numerical coefficients of the initial and final terms, viz. instead of $c \propto b^2$,

$d \infty b^3$, etc., I write $c \infty - b^2$, $d \infty 2b^3$, etc. As appears by the table these are for shortness denoted by C, D respectively, and so for weight 4, the forms are called E, E_2, for weight 5, F, F_2, for weight 6, G, G_2, G_3, G_4, and so on, the unsuffixed letters having thus an implied suffix, not 0 but 1. The table is

Table of Reductions.

$w=$			
2	$c \infty - b^2$	C	
3	$d \infty 2b^3$	D	
4	$e \infty 3c^2$	E	
	c^2 " b^4	$E_2 =$	C^2
5	$f \infty - 6bc^2$	F	
	cd " $-2b^5$	$F_2 =$	CD
6	$g \infty - 10d^2$	G	
	ce " $-c^3$	G_2	
	d^2 " $-3b^2c^2$	$G_3 =$	$CE - G_2$
	c^3 " $-b^6$	$G_4 =$	C^3
7	$h \infty 20bd^2$	H	
	cf " $3bc^3$	H_2	
	de " $6b^3c^2$	$H_3 =$	DE
	c^2d " $2b^7$	$H_4 =$	C^2D
8	$i \infty 35e^2$	I	
	cg " $2cd^2$	I_2	
	$3df$ " $10b^2d^2$	$I_3 =$	$CG - I_2$
	e^2 " $9c^4$	$I_4 =$	E^2
	c^2e " b^2c^3	$I_5 =$	CG_2
	cd^2 " $3b^4c^2$	$I_6 =$	CG_3
	c^4 " b^8	$I_7 =$	C_4
9	$j \infty - 70be^2$	J	
	$2ch$ " $-20d^3$	J_2	
	dg " $-4bcd^2$	J_3	
	ef " $-20b^3d^2$	$J_4 =$	$\frac{1}{2}(2CH - J_2 - 7J_3)$
	$2c^2f$ " $-3bc^4$	$J_5 =$	$\frac{1}{2}(EF - J_4)$
	cde " $-2b^3c^3$	$J_6 =$	DG_2
	d^3 " $-6b^5c^2$	$J_7 =$	DG_3
	c^3d " $-2b^9$	$J_8 =$	C^3D
10	$k \infty - 12bf^2$	K	
	ci " $-5ce^2$	K_2	
	$4dh$ " $-35b^2e^2$	$K_3 =$	$CI - K_2$
	$16eg$ " $-80bd^3$	K_4	
	f^2 " $-32c^2d^2$	$K_5 =$	$\frac{1}{15}(16EG - K_4)$
	c^2g " $-2b^2cd^2$	$K_6 =$	CI_2
	ce^2 " $-3c^5$	$K_7 =$	EG_2
	$3cdf$ " $-10b^4d^2$	$K_8 =$	CI_3
	d^2e " $-9b^2c^4$	$K_9 =$	EG_3
	c^3e " $-b^4c^3$	$K_{10} =$	C^2G_2
	c^2d^2 " $-3b^6c^2$	$K_{11} =$	C^2G_3
	d^5 " $-b^{10}$	$K_{12} =$	C_5
11	$l \infty 252bf^2$	L	
	$2cj$ " $35de^2$	L_2	
	di " $10bce^2$	L_3	
	eh " $20cd^3$	L_4	
	$16c^2h$ " $-70b^3e^2$	$L_5 =$	$-DI + L_3 + 2L_4$
	fg " $160b^2d^3$	$L_6 =$	$8CJ_2 - L_5$
	cef " $-2bc^2d^2$	$L_7 =$	$-\frac{1}{30}(FG - L_6)$
	cdg " $4b^3cd^2$	$L_8 =$	DI_2
	d^2f " $3bc^5$	$L_9 =$	$\frac{1}{2}(FG_2 - L_7)$
	$12c^3f$ " $-20b^5d^2$	$L_{10} =$	$-DI_3 + 3L_9$
	de^2 " $18b^3c^4$	$L_{11} =$	DE^2
	c^2de " $2b^5c^3$	$L_{12} =$	CDG_2
	cd^3 " $6b^7c^2$	$L_{13} =$	CDG_3
	c^4d " $2b^{11}$	$L_{14} =$	C^4D
12	$m \infty 462g^2$	M	
	$3ck$ " $42cf^2$	M_2	
	ei " $15e^3$	M_3	
	$15dj$ " $378b^2f^2$	$M_4 =$	$3CK - M_2$
	$25fh$ " $175bde^2$	M_5	
	g^2 " $125c^2e^2$	$M_6 =$	$\frac{1}{21}(25EI - 25M_3 - 4M_5)$
	ceg " d^4	$M_7 =$	$\frac{1}{100}(G^2 - M_6)$
	c^2i " $5b^2ce^2$	$M_8 =$	CK_2
	$5d^2g$ " $20bcd^3$	$M_9 =$	$-3GG_2 + 5EI_2 - 2M_7$
	cf^2 " $20c^3d^2$	$M_{10} =$	$GG_2 - M_7$
	$4cdh$ " $35b^4e^2$	$M_{11} =$	CK_3
	$18def$ " $80b^3d^3$	$M_{12} =$	$\frac{1}{10}(10CK_4 - 160M_7 - 32M_9 + 54M_{10})$
	e^3 " $36b^2c^2d^2$	$M_{13} =$	$\frac{1}{8}(9CK_5 - 9M_{10} - M_{12})$
	c^2e^2 " c^6	$M_{14} =$	G_2^2
	c^3g " $2b^4cd^2$	$M_{15} =$	C^2I_2
	cd^2e " $3b^2c^5$	$M_{16} =$	$EI_5 - M_{14}$
	$3c^2df$ " $10b^6d^2$	$M_{17} =$	CK_8
	d^4 " $9b^4c^4$	$M_{18} =$	G_3^2
	c^4e " $-3b^6c^3$	$M_{19} =$	C^3G_2
	c^3d^2 " $3b^8c^2$	$M_{20} =$	C^3G
	c^6 " b^{12}	M_{21}	

Where no reduction is given, the form is irreducible, i. e. it is a perpetuant.

63. As to these reductions it may be observed that in very many cases we have the sharp seminvariant given as an actual product $E_2 = C^2$, $F_2 = CD$, $G_4 = C^3$, etc. We have next other reductions such as $G_3 = CE - G_2$ where on the right-hand side there is a single product; this has a final the same as that of the seminvariant which is to be reduced, so that eliminating this term from the seminvariant and product in question we have an expression which must be a linear combination (with numerical coefficients) of the preceding seminvariants of the same weight. To take a less simple example, $L_5 = -DI + L_3 + 2L_4$; here $L_5 = -fg + 16c^2h \ldots. - 70b^3e^2$, and $DI = (d - 3bc + 2b^3)(i \ldots. + 35e^2)$ has the final $+ 70b^3e^2$. The verification is

$$\begin{array}{rcll} -DI &=& -di & \ldots. -70b^3e^2 \\ +L_3 &=& di - 2eh + fg & \\ +2L_4 &=& \quad 2eh - 2fg & \\ \hline L_5 &=& \qquad -fg & \ldots. -70b^3e^2 \end{array}$$

The only case in which we have on the right-hand side two products is $(d^2g \infty bcd^3)$, $M_9 = -3GG_2 + 5EI_2 - 2M_7$; viz. here the final of M_9 is bcd^3 which is incomposite (viz. it is not the product of two power-enders), this is in fact the first instance of a quintic seminvariant with an incomposite final and which is nevertheless reducible. For observe the next seminvariant M_{10} has the final c^3d^2, which is a product in the two ways $c^2 . cd^2$ and $c^3 . d^2$; we have thus the two products $(e \infty c^2)(cg \infty cd^2)$ and $(ce \infty c^3)(g \infty d^2)$ that is EI_2 and GG_2 with the same final c^3d^2, and combining them so as to eliminate this term we have an expression having the final bcd^3, and which is thus expressible in terms of M_9 and preceding seminvariants: the verification is

$$\begin{array}{rcll} -3GG_2 &=& -3ceg & +3d^2g \ldots. +60bcd^3 - 30c^3d^2 \\ +5EI_2 &=& +5ceg & \qquad\qquad -40bcd^3 + 30c^3d^2 \\ -2M_7 &=& -2ceg + 2cf^2 + 2d^2g & \\ \hline M_9 &=& \qquad 2cf^2 + 5d^2g \ldots. + 20bcd^3 & \end{array}$$

64. I annex to this a table (taken from the square diagrams) for the initials and finals of the sharp seminvariants for the weights 13, 14, 15, and 16.

$w=$		
13	$n \infty bg^2$	N
	cl “ df^2	N_2
	dk “ bcf^2	3
	ej “ be^3	4
	fi “ cde^2	5
	c^2j “ b^3f^2	6
	gh “ b^2de^2	7
	ceh “ bc^2e^2	8
	d^2h “ bd^4	9
	cfg “ c^3d^3	10
	cdi “ b^3ce^2	11
	deg “ b^2cd^3	12
	df^2 “ bc^3d^2	13
	c^3h “ b^5e^2	14
	e^2f “ b^4d^3	15
	c^2ef “ b^3c^2d	16
	cd^2f “ bc^6	17
	c^2dg “ b^5cd^2	18
	cde^2 “ b^3c^5	19
	c^4f “ b^7d^2	20
	d^3e “ b^5c^4	21
	c^3de “ b^7c^3	22
	c^2d^3 “ b^9c^2	23
	c^5d “ b^{13}	24
14	$o \infty h^2$	O
	cm “ cg^2	O_2
	ek “ ef^2	3
	dl “ b^2g^2	4
	fj “ bdf^2	5
	gi “ c^2f^2	6
	cei “ ce^3	7
	h^2 “ d^2e^2	8
	c^2k “ b^2cf^2	9
	d^2i “ b^2e^3	10
	cfh “ $bcde^2$	11
	cg^2 “ c^3e^2	12
	dfg “ cd^4	13
	cdj “ b^4f^2	14
	deh “ b^3de^2	15
	e^2g “ $b^2c^2e^2$	16
	c^2eg “ b^2d^4	17
	ef^2 “ bcd^3	18
	c^2f^2 “ c^4d^2	19
	c^3i “ b^4ce^2	20
	cd^2g “ b^3cd^3	21
	$cdef$ “ $b^2c^3d^2$	22
	ce^3 “ c^7	23
	c^2dh “ b^6e^2	24
	d^3f “ b^5d^3	25
	d^2e^2 “ $b^4c^2d^2$	26
	c^3e^2 “ b^2c^6	27
	d^4g “ b^6cd^2	28
	c^2d^2e “ b^4c^5	29
	c^3df “ b^8d^2	30
	cd^4 “ b^6c^4	31
	c^5e “ b^8c^3	32
	c^4d^2 “ $b^{10}c^2$	33
	c^7 “ b^{14}	34

$w=$		
15	$p \infty ch^2$	P
	cm “ dg^2	P_2
	el “ f^3	3
	dm “ bcg^2	4
	fk “ bef^2	5
	gj “ cdf^2	6
	cej “ de^3	7
	c^2l “ b^3g^2	8
	d^2j “ b^2df^2	9
	hi “ bc^2f^2	10
	cfi “ bce^3	11
	cgh “ bd^2e^2	12
	dfh “ c^2de^2	13
	e^2h “ d^5	14
	cdk “ b^3cf^2	15
	dei “ b^3e^2	16
	c^2eh “ b^2cde^2	17
	dg^2 “ bc^3e^2	18
	efg “ bcd^4	19
	c^2fg “ c^3d^3	20
	c^3j “ b^5f^2	21
	e^4h “ b^4de^2	22
	cd^2h “ $b^3c^2e^2$	23
	$cdeg$ “ b^3d^4	24
	f^3 “ $b^2c^2d^3$	25
	cdf^2 “ bc^4d^2	26
	ce^2f “ b^5ce^2	27
	d^3g “ b^4cd^3	28
	d^2ef “ $b^3c^3d^2$	29
	c^3ef “ bc^7	30
	c^4h “ b^7e^2	31
	c^2d^2f “ b^6d^3	32
	de^3 “ $b^5c^2d^2$	33
	c^2de^2 “ b^3c^6	34
	c^3dg “ b^7cd^2	35
	cd^3e “ b^5c^5	36
	c^5f “ b^9d^2	37
	d^5 “ b^7c^4	38
	c^4de “ b^9c^3	39
	c^3d^3 “ $b^{11}c^2$	40
	c^6d “ b^{15}	41

$w=$		
16	$q \infty i^2$	Q
	co “ ch^2	Q_2
	em “ eg^2	3
	dn “ b^2h^2	4
	fl “ bdg^2	5
	gk “ bf^3	6
	cek “ c^2g^2	7
	hj “ cef^2	8
	i^2 “ d^2f^2	9
	cgi “ e^4	10
	c^2m “ b^2cg^2	11
	c^3k “ b^2ef^2	12
	dej “ $bcdf^2$	13
	dfi “ bde^3	14
	e^2i “ c^3f^2	15
	ch^2 “ c^2e^3	16
	dgh “ cd^2e^2	17
	cdl “ b^4g^2	18
	dej “ b^3df^2	19
	c^2ei “ $b^2c^2f^2$	20
	efh “ b^2ce^3	21
	eg^2 “ $b^2d^2e^2$	22
	c^2fh “ bc^2de^2	23
	c^2g^2 “ bd^5	24
	f^2g “ c^4e^2	25
	ce^2g “ c^2d^4	26
	c^3k “ b^4cf^2	27
	cd^2i “ b^4e^3	28
	$cdeh$ “ b^3cde^2	29
	$cdfg$ “ $b^2c^3e^2$	30
	d^2eg “ b^2cd^4	31
	cef^2 “ bc^3d^3	32
	d^2f^2 “ c^5d^2	33
	c^2dj “ b^6f^2	34
	d^3h “ b^5de^2	35
	c^3ej “ $b^4c^2e^2$	36
	c^3f^2 “ b^4d^4	37
	de^2f “ $b^3c^2d^3$	38
	e^4 “ $b^2c^4d^2$	39
	c^2e^3 “ c^8	40
	c^4i “ b^6ce^2	41
	c^2d^2g “ b^5cd^3	42
	c^2def “ $b^4c^3d^2$	43
	cd^2e^2 “ b^2c^7	44
	c^3dh “ b^8e^2	45
	cd^3f “ b^7d^3	46
	c^4e^2 “ $b^6c^2d^2$	47
	d^4e “ b^4c^6	48
	c^5g “ b^8cd^2	49
	c^3d^2e “ b^6c^5	50
	c^4df “ $b^{10}d^2$	51
	c^2d^4 “ b^8c^4	52
	c^6e “ $b^{10}c^3$	53
	c^5d^2 “ $b^{12}c^2$	54
	c^8 “ b^{16}	55

It would be interesting to complete this into a table of reductions as given for the weights 2 to 12.

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65. We can by means hereof establish in regard to the specific blunt seminvariants, a general theory of reduction, or say a theory of the relations which exist between the seminvariants of a given degree and the powers and products of seminvariants of inferior degrees. To exhibit the form of these it will be sufficient to take Ω a sum of two parts, $=\Omega'+\Omega''$, but the more general assumption is Ω a sum of any number of parts, $=\Omega'+\Omega''+\Omega''' \dots$ Taking then $\Omega=\Omega'+\Omega''$, where for the Ω' and Ω'' separately the sum of the $(x, y, z \dots)$ is $=0$, suppose that to the $(0, C, D, E, \dots)$ of Ω there correspond $(0, C', D', E', \dots)$ for Ω' and $(0, C'', D'', E'', \dots)$ for Ω''. We have

$$\begin{aligned}
C &= C'+C'',\\
D &= D'+D'',\\
E &= E'+E''+C'C'',\\
F &= F'+F''+C'D''+C''D',\\
G &= G'+G''+C'E''+C''E'+D'D'',\\
&\vdots
\end{aligned}$$

the law of which is obvious.

66. We have for instance

$\Omega^4=(\Omega'+\Omega'')^4, =\Omega'^4+6\Omega'^2\Omega''^2+\Omega''^4$ (since $\Omega'=0$, $\Omega''=0$), that is

$$\begin{aligned}
&(C'+C'')^2c^2 = && C'^2c^2+6\,C'b^2\,.\,C''b^2+C''^2c^2\\
&+(E'+E''+C'C'')b^4 && +E'b^4 \qquad\qquad\qquad +E''b^4
\end{aligned}$$

where, and in what follows, c^2, b^4, b^2 are for shortness written instead of $[c^2]$, $[b^4]$, $[b^2]$ to denote the specific blunt seminvariants ending in c^2, b^4, b^2 respectively.

The terms in C'^2, C''^2, E', E'' are identical on each side of the equation and destroy each other: omitting these we have only the terms in $C'C''$ which must be equivalent on the two sides of the equation, and comparing coefficients we find the relation

$$2c^2+b^4=6\,.\,b^2\,.\,b^2$$

which of course means $2[c^2]+[b^4]=6[b^2][b^2]$, viz. this is

$$2(2e-8bd+6c^2)+(-4e+16bd+12c^2-48b^2c+24b^4)=6(-2c+2b^2)^2.$$

In like manner for $\Omega^6, = (\Omega' + \Omega'')^6$ we have

$$\begin{array}{ll} (C'+C'')^3 & .\,d^2 \\ +(D'+D'')^2 & .\,c^3 \\ +(C'+C'')(E'+E''+C'C'') & .\,b^2c^2 \\ +(G'+G''+C'E''+C''E'+D'D'') & .\,b^6 \end{array}$$

equal to

$$\left\{\begin{array}{ll} C'^3 & .\,d^2 \\ +D'^2 & .\,c^3 \\ +C'E' & .\,b^2c^2 \\ +G' & .\,b^6 \end{array}\right\} + 15\left\{\begin{array}{l} C'^2.\,c^2 \\ +E'.\,b^4 \end{array}\right\} C''.\,b^2 + 20D'.\,b^3.\,D''.\,b^3 + 15\,C'^2.\,b^2\left\{\begin{array}{l} C''^2.\,c^2 \\ +E''.\,b^4 \end{array}\right\} + \left\{\begin{array}{ll} C''^3 & .\,d^2 \\ +D''^2 & .\,c^3 \\ +C''E'' & .\,b^2c^2 \\ +G'' & .\,b^6 \end{array}\right\}$$

Here omitting the terms which destroy each other and comparing the coefficients of the remaining terms, viz. $C'^2C''+C''^2C'$, $D'D''$ and $C'E''+C''E'$ we find the relations

$$\begin{aligned} 3d^2 + b^2c^2 &= 15\,.\,c^2\,.\,b^2 \\ 2c^3 + b^6 &= 20\,.\,b^3\,.\,b^3 \\ b^2c^2 + b^6 &= 15\,.\,b^4\,.\,b^2 \end{aligned}$$

which may be easily verified. There are on the right-hand side only products of two parts, but this is on account of the special assumption $\Omega = \Omega' + \Omega''$, a sum of two parts.

67. I write now

$$\begin{array}{ll} \Omega_2 = \alpha x + \beta y & ,\ S_2x = 0, \\ \Omega_3 = \alpha x + \beta y + \gamma z & ,\ S_3x = 0, \\ \Omega_4 = \alpha x + \beta y + \gamma z + \delta w & ,\ S_4x = 0, \\ \Omega_5 = \alpha x + \beta y + \gamma z + \delta w + \varepsilon t & ,\ S_5x = 0, \\ \Omega_6 = \alpha x + \beta y + \gamma z + \delta w + \varepsilon t + \zeta u, & S_6x = 0, \\ \vdots & \end{array}$$

and I say that Ω_2 and Ω_3 cannot break up: but that Ω_4 breaks up if it becomes a sum of $2 + 2$ terms (i. e. a sum of two parts Ω_2 for each of which $S_2x = 0$, and so in other cases): that Ω_5 breaks up if it becomes a sum of $2 + 3$ terms, Ω_6 breaks up if it becomes a sum of $2 + 4$ or $2 + 2 + 2$ terms, or if it becomes a sum of $3 + 3$ terms: and similarly for any higher suffix.

The condition that Ω_4 may break up is $x + y = 0$, $x + z = 0$, or $y + z = 0$, or what is the same thing it is $\Pi_3(x + y) = 0$, where $\Pi_3(x + y)$ is the product of

the three sums each containing x; this is a symmetric function, we in fact have $\Pi_3(x+y)=x^3+x^2(y+z+w)+x(yz+yw+zw)+yzw, =xyz+xyw+xzw+yzw, =-D$.

The condition in order that Ω_5 may break up is $x+y=0, \ldots$ or $w+t=0$, say this is $\Pi_{10}(x+y)=0$, where $\Pi_{10}(x+y)$ denotes the product of the ten sums $x+y, \ldots, w+t$. It will be shown that we have $\Pi_{10}(x+y)=-D^2E+CDF-F^2$.

The condition in order that Ω_6 may break up is, $x+y=0, \ldots$ or $t+u=0$, or again if $x+y+z=0, \ldots$ or $x+t+u=0$, viz. it is $\Pi_{15}(x+y)\Pi_{10}(x+y+z)=0$, where $\Pi_{15}(x+y)$ is the product of the fifteen sums $x+y, \ldots, t+u$, and $\Pi_{10}(x+y+z)$ is the product of the ten sums $x+y+z, \ldots x+t+u$, each containing x: $\Pi_{15}(x+y)$ and $\Pi_{10}(x+y+z)$ are symmetric functions, the expressions for which will be given further on: the weights in the capital letters are 15 and 10 respectively. And similarly for Ω with any higher suffix, we have the condition that this may break up.

I introduce the factors $\Pi_4x=E$, $\Pi_5x=-F$, $\Pi_6x=G, \ldots$ respectively and write for

$$
\begin{array}{ll}
\Omega_4 & M_7=\Pi_4x\Pi_3(x+y)=-DE \text{ as above,}\\
\Omega_5 & M_{15}=\Pi_5x\Pi_{10}(x+y)=-F(-D^2E+CDF-F^2) \text{ as above,}\\
\Omega_6 & M_{31}=\Pi_6x\Pi_{15}(x+y)\Pi_{10}(x+y+z),\\
 & \vdots
\end{array}
$$

where observe that for the even suffixes of Ω, the last factors $\Pi_3(x+y)$, $\Pi_{10}(x+y+z), \ldots$ denote the products of the sums $x+y, x+y+z, \ldots$ which contain x, that is in each case the products of only half the whole number of such linear factors. The suffixes of M show the weights in the capital letters $C, D, E, F, G, \ldots$ viz. these are $4+3, =7$: $5+10, =15$, $6+15+10, =31$, and so on; the law is obvious, and for Ω_n the weight is $=2^{n-1}-1$.

68. To explain the Strohian theory of perpetuants, I assume explicitly as presently appears. For perpetuants of any given degree δ, we consider in $\Omega_\delta^w (w=\delta$ at least) the terms containing seminvariants of the given degree: for instance $\delta=4$, $w=12$ these are

$$
\begin{array}{ll}
 & C^4E \;.\; b^2f^2\\
+ & CD^2E \,.\, bde^2\\
+ & C^2E^2 \;.\; c^2e^2\\
+ & E^3 \;\;.\; d^4
\end{array}
$$

where the capital expressions all contain as factor the letter E of the weight 4. By making Ω to break up it is assumed that *we obtain all the reductions of the*

seminvariants of the degree and weight in question; and every such seminvariant, if it be reducible, will be reduced by means of the resulting formulæ. Now there are seminvariants which are not reducible by these formulæ: in the example just considered, the seminvariant bde^2 has the coefficient CD^2E which contains the factor DE, $=xyzw(x+y)(x+z)(x+w)$ which vanishes when Ω_4 breaks up; so that supposing Ω_4 to break up, the seminvariant bde^2 disappears from the formulæ, and we have no reduction of this seminvariant. And again it is assumed *that every seminvariant which does not in this way disappear from the equation is reducible.* The irreducible seminvariants are thus the seminvariants which when Ω breaks up into a sum of two or more parts disappear from the formulæ; viz. the seminvariants which thus disappear are the perpetuants.

69. In the case considered of quartic seminvariants it has just been seen that, for the weight 12, bde^2 is a perpetuant; and so in general for the weight w, every quartic seminvariant multiplied into a product of capitals which contains the factor DE is a perpetuant: for the weight 7 the only term is $DE\,.\,bc^3$, viz. the product of capitals is here $=DE$; and for any higher weight w we have products which are equal to DE into products of the weight $w-7$ in C, D, E: and we thus see that the *G. F.* for quartic perpetuants is $=x^7 \div 2\cdot3\cdot4$.

70. For quintic perpetuants we consider in $\Omega_5^w(w=5$ at least) the terms which contain quintic perpetuants, for instance $w=15$ the terms are

$$\begin{array}{ll}
C^5F & .\,b^3g^2 \\
+\,C^2D^2F & .\,b^2df^2 \\
+\,C^3EF & .\,bc^2f^2 \\
+\,D^2EF & .\,bce^3 \\
+\,CE^2F & .\,bd^2e^2 \\
+\,CDF^2 & .\,c^2de^2 \\
+\,F^3 & .\,d^5
\end{array}$$

where the functions of the capitals all contain the factor F; the finals b^3g^2, b^2df^2, ... are arranged in AO. Supposing Ω_5 to break up, we have an expression M, $=-D^2EF+CDF^2-F^3$, which is $=0$, and using this value of M to eliminate the term D^2EF which belongs to the seminvariant bce^3 the final whereof is highest in AO, viz. writing $D^2EF=-M+CDF^2-F^3$ the expression is

$$\begin{array}{rl} C^5F & .\, b^3g^2 \\ +C^2D^2F & .\, b^2df^2 \\ +C^3EF & .\, bc^2f^2 \\ +(-M+CDF^2-F^3) & .\, bce^3 \\ +CE^2F & .\, bd^2e^2 \\ +CDF^2 & .\, c^3de^2 \\ +F^3 & .\, d^5 \end{array} \quad \text{that is} \quad \begin{array}{ll} C^5F & .\, b^3g^2 \\ +C^2D^2F & .\, b^2df^2 \\ +C^3EF & .\, bc^2f^2 \\ -M & .\, bce^3 \\ +CE^2F & .\, bd^2e^2 \\ +CDF^2 & .\, (c^2de^2+bce^3) \\ +F^3 & .\, (d^5-bce^3) \end{array}$$

and here when Ω_5 breaks up we have $M=0$, that is the seminvariant bce^3 disappears from the equation, and it is thus a perpetuant: but b^3g^2, b^2df^2, bc^2f^2 and the combinations $c^2de^2+bce^3$, and d^5-bce^3 are severally reducible.

The degree 15 is evidently the lowest degree for which there is an irreducible quintic seminvariant, and for any higher weight w the number of such seminvariants is equal to the number of capital terms which have the factor D^2EF, viz. this is equal to the number of terms weight $w-15$ which can be made up with C, D, E, F; and hence

$$\text{for quintic perpetuants } G.\,F. = x^{15} \div 2\,.\,3\,.\,4\,.\,5\,.$$

71. For the degree 6, $M=\Pi_6 x\Pi_{15}(x+y)\Pi_{10}(x+y+z)$ is a function of the capitals of the weight 31, and we thence at once infer that

$$\text{for sextic perpetuants } G.\,F. = x^{31} \div 2\,.\,3\,.\,4\,.\,5\,.\,6\,.$$

But it is worth while to write down the expression for M: I do this annexing to each term the seminvariant (i. e. final term) which belongs to it, arranging these final terms in AO; the value thus arranged is

$M=$		finals in AO
$+1$	D^4E^2FG	$bcei^3$
-2	CD^3EF^2G	$bdehi^2$
$+1$	$C^2D^2F^3G$	be^2gi^2
$+2$	D^2EF^3G	$befh^3$
-2	CDF^4G	bf^2gh^2
$+1$	F^5G	bg^5
-1	D^5EG^2	c^2di^3
$+1$	CD^4FG^2	cd^2hi^2
$+1$	$C^2D^2EFG^2$	$cdegi^2$
-4	$D^2E^2FG^2$	$cdfh^3$
-1	$C^3DF^2G^2$	ce^2fi^2
-1	$D^3F^2G^2$	ce^2h^3
$+4$	$CDEF^2G^2$	$cefgh^2$
$+1$	$C^2F^3G^2$	cf^3h^2
$+4$	EF^3G^2	cfg^4
-1	$C^2D^3G^3$	d^3gi^2
$+4$	D^3EG^2	d^2eh^3

It thus appears that the single sextic perpetuant of the weight 31 is $bcei^3$, and generally that for any higher weight the sextic perpetuants are such that the conjugate capital terms contain each of them the factor D^4E^2FG.

The like reasoning shows that

for perpetuants of degree n, $G.\,F.$ is $= x^{2^{n-1}-1} \div 2\,.\,3\,.\,4\,\ldots\,.\,n$.

Investigation of the Values of the Foregoing Functions $\Pi_{10}(x+y)$, $\Pi_{15}(x+y)$ *and* $\Pi_{10}(x+y+z)$. *Art. Nos.* 72 *to* 74.

72. If x, y, z, w, t are the roots of a quintic equation, say

$$\lambda - x\,.\,\lambda - y\,.\,\lambda - z\,.\,\lambda - w\,.\,\lambda - t = (1,\ B,\ C,\ D,\ E,\ F\between\lambda,\ 1)^5 = 0$$

we require the product $\Pi_{10}(x+y)$, of the sum of two roots in the particular case $B=0$. But in order to the determination of the expression for $\Pi_{10}(x+y+z)$, we require the value of $\Pi_{10}(x+y)$ in the general case, B any value whatever.

Writing

$$x = -\tfrac{1}{2}(\theta+\omega),$$
$$y = -\tfrac{1}{2}(\theta+\omega),$$

and therefore

$$\theta + x + y = 0,$$

we have

$$(\theta+\omega)^5 - 2B(\theta+\omega)^4 + 4C(\theta+\omega)^3 - 8D(\theta+\omega)^2 + 16E(\theta+\omega) - 32F = 0,$$

and the like equation with $-\omega$ for ω. Hence writing $\omega^2 = M$, we have

$$(\theta^5 - 2B\theta^4 + 4C\theta^3 - 8D\theta^2 + 16E\theta - 32F) + M(10\theta^3 - 12B\theta^2 + 12C\theta - 8D) + M^2(5\theta - 2B) = 0,$$

$$(5\theta^4 - 8B\theta^3 + 12C\theta^2 - 16D\theta + 16E) + M(10\theta^2 - 8B\theta + 4C) + M^2.1 = 0,$$

which are of the form $A + BM + CM^2 = 0$, $A' + B'M + C'M^2 = 0$ and give therefore by elimination of M the equation

$$-(CA' - C'A)^2 + (BC' - B'C)(AB' - A'B) = 0\,;$$

the left-hand side is here a function of θ of the degree 10 vanishing when $\theta + x + y = 0$, and which must therefore be, save as to a numerical factor, the product $\Pi_{10}(\theta + x + y)$. And we thus find

$$\Pi_{10}(\theta + x + y) =$$

$$-\left\{24\theta^5 - 48B\theta^4 + \begin{pmatrix} 56C \\ +16B^2 \end{pmatrix}\theta^3 + \begin{pmatrix} -72D \\ -24BC \end{pmatrix}\theta^2 + \begin{pmatrix} 64E \\ +32BD \end{pmatrix}\theta + \begin{pmatrix} -32BE \\ +32F \end{pmatrix}\right\}^2$$

$$+\left\{40\theta^3 - 48B\theta^2 + \begin{pmatrix} 8C \\ +16B^2 \end{pmatrix}\theta + \begin{pmatrix} 8D \\ -8BC \end{pmatrix}\right\}\cdot\left\{40\theta^7 - 112B\theta^6 + \begin{pmatrix} 136C \\ +80B^2 \end{pmatrix}\theta^5 + \right.$$

$$\left.\begin{pmatrix} -120D \\ -200BC \end{pmatrix}\theta^4 + \begin{pmatrix} 0E \\ +192BD \\ +128C^2 \end{pmatrix}\theta^3 + \begin{pmatrix} 320F \\ -64BE \\ -526CD \end{pmatrix}\theta^2 + \begin{pmatrix} -256BF \\ +128CE \\ -128D^2 \end{pmatrix}\theta + \begin{pmatrix} 128CF \\ -128DE \end{pmatrix}\right\},$$

which is

$$= 1024\theta^{10} \ldots\ldots + 1024(-F^2 + CDF + 2BEF - BC^2F - D^2E + BCDE),$$

and which therefore for $B = 0$ gives

$$\Pi_{10}(x+y) = -F^2 + CDF - D^2E.$$

73. Suppose now x, y, z, w, t, u are the roots of a sextic equation, say

$$\lambda - x \,.\, \lambda - y \,.\, \lambda - z \,.\, \lambda - w \,.\, \lambda - t \,.\, \lambda - u = (1, B, C, D, E, F, G \between \lambda, 1)^6 = 0\,.$$

Considering here the product $\Pi_{20}(x+y+z)$ of the sums of 3 roots, if $B = 0$, this will be a perfect square (for each sum $x+y+z$ is equal to $-$ a sum $(w+t+u)$) say it is the square of $\Pi_{10}(x+y+z)$, where the $x+y+z$ refers to the ten sums each containing x, and we wish to find this function $\Pi_{10}(x+y+z)$. Writing for the equation whose roots are y, z, w, t, u,

$$\lambda - y \,.\, \lambda - z \,.\, \lambda - w \,.\, \lambda - t \,.\, \lambda - u = (1, B', C', D', E', F' \between \lambda, 1)^5,$$

we have by what precedes $\Pi_{10}(\theta + y + z) =$ a function $(* \between \theta, 1)^{10}$, viz. this is the above-mentioned function with B', C', D', E', F' in place of the unaccented letters. Introducing a new root x and for λ writing as we may do θ, we have

$$\begin{aligned}\theta - x \,.\, \theta - y \,.\, \theta - z \,.\, \theta - w \,.\, \theta - t \,.\, \theta - u &= (\theta - x)\,.\,(1, B', C', D', E', F' \between \theta, 1)^5 \\ &= (1, B, C, D, E, F, G \between \theta, 1)^6;\end{aligned}$$

that is we have

$$\begin{array}{ll} B = B' - \theta \text{ or conversely} & B' = B + \theta, \\ C = C' - B'\theta & C' = C + B\theta + \theta^2, \\ D = D' - C'\theta & D' = D + C\theta + B\theta^2 + \theta^3, \\ E = E' - D'\theta & E' = E + D\theta + C\theta^2 + B\theta^3 + \theta^4, \\ F = F - E'\theta & F' = E + E\theta + D\theta^2 + C\theta^3 + B\theta^4 + \theta^5, = -\dfrac{G}{\theta}, \\ G = \quad - F'\theta & \end{array}$$

where I have retained B, but the value hereof is in fact $= 0$. In the foregoing function $(* \between \theta, 1)^{10}$ with the accented letters, writing for these their values $B' = \theta$, $C' = C + \theta^2$, $D' = D + C\theta + \theta^3$, etc., which belong to $B = 0$, we find

$$\begin{aligned}1024\Pi_{10}(\theta + y + z) = &-(48\theta^5 + 56C\theta^3 + 24D\theta^2 + 64E\theta + 32F)^2 \\ &+(16\theta^3 + 8C\theta + D)\{144\theta^7 + 264C\theta^5 + 72D\theta^4 + 128(C^2+E)\theta^3 \\ &+ 192F\theta^2 + 128CE\theta + 128(CF - DE)\},\end{aligned}$$

which equation divides by 64. Writing herein $\theta = x$, we have

$$\begin{aligned}16\Pi_{10}(x+y+z) = &-(6x^5 + 7Cx^3 + 3Dx^2 + 8Ex + 4F)^2 \\ &+(2x^3 + Cx + D)\{18x^7 + 33Cx^5 + 9Dx^4 + \\ &16(C^2+E)x^3 + 24Fx^2 + 16CEx + 16(CF - DE)\}\end{aligned}$$

where $x^6+Cx^4+Dx^3+Ex^2+Fx+G$ is $=0$: the value ought in virtue of this equation to reduce itself to a mere function of the coefficients, and we in fact find that the equation is

$$16\Pi_{10}(x+y+z)=$$
$$(16C^2-64E)(x^6+Cx^4+Dx^3+Ex^2+Fx)+16CDF-16D^2E-16F^2,$$

reducing itself to

$$-(16C^2-64E)G \qquad\qquad +16CDF-16D^2E-16F^2,$$

viz. dividing each side by 16, we have

$$\Pi_{10}(x+y+z)=4EG-C^2G-F^2+CDF-D^2E,$$

which is the required result. The equation $(\theta^2-1)^3=0$, for which $x, y, z, w, t, u=1, 1, 1, -1, -1, -1$ gives a numerical verification.

74. I find also, for the same value $B=0$, the function $\Pi_{15}(x+y)$. Writing as before

$$x=-\tfrac{1}{2}(\theta+\omega),$$
$$y=-\tfrac{1}{2}(\theta-\omega),$$

and therefore

$$\theta+x+y=0,$$

we have

$$(\theta+\omega)^6+4C(\theta+\omega)^4-8D(\theta+\omega)^3+16E(\theta+\omega)^2-32F(\theta+\omega)+64G=0,$$

and the like equation with $-\omega$ for ω. Hence writing $\omega^2=M$, we have

$$(\theta^6+4C\theta^4-8D\theta^3+16E\theta^2-32F\theta+64G)+M(15\theta^4+24C\theta^2-24D\theta+16E)$$
$$+M^2(15\theta^2+4C)+M^3=0,$$
$$(6\theta^5+16C\theta^3-24D\theta^2+32E\theta-32F)+M(20\theta^3+16C\theta-8D)+M^2.6\theta=0,$$

say these equations are $aM^3+bM^2+cM+d=0$, $pM^2+qM+r=0$. Eliminating M we have

$$\begin{array}{ll}
a^2.r^3 & a=1,\\
-ab.qr^2 & b=15\theta^2+4C,\\
+ac(-2pr^2+q^2r) & c=15\theta^4+24C\theta^2-24D\theta+16E,\\
+b^2.pr^2 & d=\theta^6+4C\theta^4-8D\theta^3+16E\theta^2-32F\theta+64G,\\
+ad(3pqr-q^3) & \\
+bc(-pqr) & p=6\theta,\\
+bd(-2p^2r+pq^2) & q=20\theta^3+16C\theta-8D,\\
+c^2.p^2r & r=6\theta^5+16C\theta^3-24D\theta^2+32E\theta-32F,\\
-cd.p^2q & \\
+d^2.p^3=0 &
\end{array}$$

The equation as far as I have calculated it is

$$-327680^{15} \ldots . -32768(-D^3G+F^3-CDF^2+D^2EF)=0;$$

the left-hand side is here $= -32768\Pi_{15}(x+y)$; and we have therefore

$$\Pi_{15}(x+y) = -D^3G+F^3-CDF^2+D^2EF,$$

the required result. It may be remarked that writing $G=0$, and throwing out a factor $-F$, we have $-F^2+CDF-D^2E$, which is the expression for $\Pi_{10}(x+y)$ in the quintic equation.

We have

$$\Pi_6 x\Pi_{15}(x+y)\Pi_{10}(x+y+z)=$$
$$G\{-D^3G+(F^2-CDF+D^2E)F\}\{(4E-C^2)G-F^2+CDF-D^2E\},$$

the developed expression whereof is the foregoing value

$$M = D^4E^2FG-2CD^3EF^2G+\text{ etc., } \textit{ante} \text{ No. 71.}$$

The Operators $P-\delta b$ and $Q-2\omega b$. Art. Nos. 75 *to* 84.

75. The analogous theory for nonunitariants is established, *ante* Nos. 24 *et seq.* For seminvariants we have

$$P = b\partial_a + c\partial_b + d\partial_c + \ldots,$$
$$Q = c\partial_b + 2d\partial_c + \ldots$$

or more definitely if the seminvariant operated upon be of the degree δ, the weight ω and extent σ, say its highest letter is a_σ, $=p$, then

$$P = b\partial_a + c\partial_b + d\partial_c \ldots + q\partial_p,$$
$$Q = \qquad c\partial_b + 2d\partial_c \ldots + \sigma_q\partial_p,$$

then we have

$$P-\delta b,\ Q-2\omega b,$$

operators each of them of the deg. weight 1.1, viz. each of them operating upon a seminvariant S of the deg. weight $\delta . \omega$ gives a seminvariant S' of the deg. weight $\delta+1 . \omega+1$; moreover, a new letter q is introduced, or say the extent is increased from σ to $\sigma+1$. For the proof it is only necessary to show that $\Delta(P-b\delta)S$ and $\Delta(Q-2b\omega)$ are each $=0$, but it is unnecessary to do this, as the like proof has already been given for nonunitariants.

The two seminvariant operators were first considered in my paper "On a Theorem Relating to Seminvariants," Quart. Math. Jour. t. XX (1885), pp. 212–213.

76. We may instead of $P - \delta b$ and $Q - 2\omega b$, consider the linear combination $Y = 2\omega(P - \delta b) - \delta(Q - 2\omega b)$, that is $2\omega P - \delta Q$, which is of deg. weight $0 . 1$, viz. it leaves the degree unaltered, while increasing as before the weight, and also the extent, each by unity. And again the combination

$$Z = \sigma(P - \delta b) - (Q - 2\omega b), \text{ that is } \sigma P - Q - (\sigma\delta - 2\omega)b,$$

where observe that $\sigma P - Q$, $= \sigma b\partial_a + (\sigma - 1)c\partial_b \ldots . + 1p\partial_0$ does not contain the new letter q, the operator Z is thus of the deg. weight $1 . 1$ increasing the degree and also the weight each by unity, but leaving the extent unaltered.

There is a special case which it is important to attend to, we may have $\sigma\delta - 2\omega = 0$, viz. this is the case when the seminvariant operated upon is in regard to the letters comprised therein an invariant. Here the two combinations Y, Z are equivalent to each other, each of them is $= \sigma b\partial_a + (\sigma - 1)c\partial_b \ldots . + 1p\partial_0$, which is an annihilator of the seminvariant (invariant) operated upon. Hence in this case we cannot replace the original forms by the linear combinations, but must retain one (no matter which) of the original forms $P - \delta b$, $Q - 2\omega b$.

77. We can by means of the foregoing operators starting from the quadric seminvariants $c - b^2$, etc., derive in order the seminvariants for the successive weights 3, 4, 5, ...

Thus writing down the series of finals (in AO as before)

b^2,	b^3,	c^2,	bc^2,	d^2,	bd^2,	e^2,	etc.
		b^4	b^5	c^3	bc^3	cd^2	
				b^2c^2	b^3c^2	b^2d^2	
				b^6	b^7	c^4	
						b^2c^3	
						b^4c^2	
						b^8	

I proceed as follows, observing, however, that when the function operated upon is an invariant seminvariant we must instead of Z write $P - \delta b$.

b^2 emerges,	$b^3 = Zb^2$,	c^2 emerges,	$bc^2 = Zc^2$,	d^2 emerges,	$bd^2 = Zd^2$,	e^2 emerges,
		$b^4 = Zb^3$	$b^5 = Zb^4$	$c^3 = Ybc^2$	$bc^3 = Zc^3$	$cd^2 = Ybd^2$
				$b^2c^2 = Zbc^2$	$b^3c^2 = Zb^2c^2$	$b^2d^2 = Zbd^2$
				$b^6 = Zb^5$	$b^7 = Zb^6$	$c^4 = Ybc^3$
						$b^2c^3 = Zbc^3$
						$b^4c^2 = Zb^3c^2$
						$b^8 = Zb^7$

viz. whenever the seminvariant to be obtained has a final containing b it is obtained by means of the operator Z (or it may be $P-\delta b$), but when there is no b then by the operator Y.

The seminvariants operated upon may be blunt or sharp, but there is an advantage in operating on the sharp forms as these are more simple and we thereby obtain for the next superior weight forms more nearly approximating to the sharp forms. We do not however by thus operating on a sharp form obtain directly a sharp form; to do this the form obtained must be modified by adding thereto a numerical multiple or multiples of a preceding sharp form: and thus the theory does not determine beforehand the forms of the sharp seminvariants. But making at each step the necessary modification (if any) we have thereby, when the sharp seminvariants of the next preceding weight are known, a very convenient process for the calculation of the sharp seminvariants of any given weight, in the AO arrangement of their final terms. Thus for the weight 10; $k \infty f^2$ is taken to be known, the next two forms $ci \infty ce^2$ and $dh \infty b^2e^2$ are calculated each from $j \infty be^2$, the expression for which is $= j - 9bi + 20ch - 28dg + 14ef + 16b^2h - 56bcg + 112bdf - 70be^2$. We have for $j \infty be^2$, $\delta = 3$, $\omega = 9$, $\sigma = 9$ and therefore

$$\tfrac{1}{3}Y = 6b\partial_a + 5c\partial_b + 4d\partial_c + 3e\partial_d + 2f\partial_e + g\partial f - i\partial_h - 2j\partial_i - 3k\partial_j,$$
$$Z = 9b\partial_a + 8c\partial_b + 7d\partial_c + 6e\partial_d + 5f\partial_e + 4g\partial_f + 3h\partial_g + 2i\partial_h + j\partial_i - 9b.$$

78. I exhibit the calculation as follows:

$\tfrac{1}{3}Y(j \infty be^2)$

		1	2	3	4	5	6	7	8	9	†	÷70	*
k	+ 1									−3	− 3+ 3	0	
bj	− 10	+ 12							+18		+ 30− 30	0	
ci	+ 45		− 45					−20			− 65+135	+ 70	+1
dh	−120			+ 80							+ 80−360	−280	−4
eg	+210				− 84		+ 14				− 70+630	+560	+8
f^2	−126					+ 28					+ 28−378	−350	−5
b^2i		- 54						−16			− 70	− 70	−1
bch		+ 0	+160								+280	+280	+4
bdg		−1		−224			+112				−280	−280	−4
bef		+ 84			+336	−280					+140	+140	+2
c^2g			−280								−280	−280	−4
cdf			+560								+560	+560	+8
ce^2			−350								−350	−350	−5
d^2e													
b^3h													
b^2cg													
b^2df													
b^2e^2													
	±256												±23

$Z(j \infty be^2)$

	1	2	3	4	5	6	7	8	9	10	†	÷ −18	*
k													
bj	+ 18								−9	−9	0	0	
ci		− 72						+40			− 32+ 32	0	
dh			+140				− 84				+ 56−128	− 72	+ 4
eg				−168		+ 56					− 112+256	+ 144	− 8
f^2					+ 70						+ 70−160	− 90	+ 5
b^2i	− 81							+32		+ 81	+ 32− 32	0	0
bch	+180	+256					−168			− 180	+ 88+128	+ 216	−12
bdg	−252		−392			+448				+ 252	+ 56−128	− 72	+ 4
bef	+126			+672	−700					− 126	− 28+ 64	+ 36	− 2
c^2g		−448									− 448−128	− 576	+32
cdf		+896									+ 896+256	+1152	−64
ce^2		−560									− 560−160	− 720	+40
d^2e													
b^3h										− 144	− 144	− 144	+ 8
b^2cg										+ 504	+ 504	+ 504	−28
b^2df										−1008	−1008	−1008	+56
b^2e^2										+ 630	+ 630	+ 630	−35

±149

The numbers (1, 2, ... 9) and (1, 2, ... 10) at the head of the columns refer to the nine terms $6b\partial_a$, $5c\partial_b$, ... of $\frac{1}{3}Y$, and the ten terms $9b\partial_a$, $8c\partial_b$, ... of Z respectively, these several operations being performed on $(j \infty be^2)$ the value of which is given above: the daggers † denote the additions which have to be made in order to obtain the proper initial term, viz. for the first † the added term is $+ 3(k \infty f^2)$ and for the second † the added term is $+ 32(ci \infty ce^2)$: the headings ÷ 70 and ÷ − 18 explain themselves, and the columns headed with an asterisk * give the results, viz. the first of these is $(ci \infty ce^2)$ and the second of them is $(dh \infty b^2e^2)$. As appears above, the value of the first of these is used in the second † column for obtaining that of the second of them.

79. We may operate with $P - \delta b$ and $Q - 2\omega b$ on a product (deg. weight $\delta.\omega$) ST of two seminvariants S, T, deg. weights $\delta'.\,\omega'$ and $\delta''.\,\omega''$ respectively, $\delta = \delta' + \delta''$, $\omega = \omega' + \omega''$. We have

$$(P - \delta b)\,ST = S.PT + T.PS - (\delta' + \delta'')\,bST, = S(P - \delta''b)\,T + T(P - \delta'b)\,S,$$

where $(P - \delta'b)\,S$ and $(P - \delta''b)\,T$ are each of them a seminvariant. And similarly

$$(Q - 2\omega b)\,ST = S.QT + T.QS - 2(\omega' + \omega'')\,bST = S(Q - 2\omega''b)\,T + T(Q - 2\omega'b)\,S,$$

where $(Q - 2\omega'b)\,S$ and $(Q - 2\omega''b)\,T$ are each of them a seminvariant. That is, operating either with $P - \delta b$ or $Q - 2\omega b$ on a product we have a sum of products;

and therefore also operating upon a sum of products (each product being of the deg. weight $\omega.\delta$) we have a sum of products, each product in such sum being of the deg. weight $\omega+1.\delta+1$, and moreover of the extent $\sigma+1$. And instead of binary products, we may, it is clear, consider ternary, quaternary, etc., products.

The like theorem applies to the derived operators Y and Z, but as to Y there is a specialty to be noticed. We have

$$\begin{aligned} Y.ST &= 2\omega(P-\delta b)\,ST-\delta\,(Q-2\omega b)\,ST,\\ &= 2\omega\{S(P-\delta''b)\,T+T(P-\delta'b)S\}-\delta\{S(Q-2\omega''b)\,T+T(Q-2\omega'b)\,S\},\\ &= S\{2\omega\,(P-\delta''b)\,T-\delta(Q-2\omega''b)\,T\}+T\{2\omega\,(P-\delta'b)\,T-\delta\,(Q-2\omega'b)\,S\}, \end{aligned}$$

where the whole of the right-hand side as being equal to $Y.ST$ is of the degree δ, but except in the particular case $\left(\frac{\delta}{\omega}=\frac{\delta'}{\omega'}=\frac{\delta''}{\omega''}\right)$ the separate products $S\{\ \}$ and $T\{\ \}$ which occur on the right-hand side are each of them of the degree $\delta+1$.

It is scarcely necessary, but it may be proper to remark that we frequently combine by addition a seminvariant S of the deg. weight $\delta.\omega$ with a seminvariant T deg. weight $\delta-\varepsilon.\omega$ of the same weight but of an inferior degree, but when this is done we regard the T as standing for $a^\varepsilon T$, and as being thus of the same deg. weight $\delta.\omega$. We have

$$(P-\delta b)\,a^\varepsilon T = a^\varepsilon PT + TPa^\varepsilon - (\varepsilon+\delta-\varepsilon)\,ba^\varepsilon T, = a^\varepsilon\{P-(\delta-\varepsilon)\,b\}\,T + T(P-\varepsilon b)\,a^\varepsilon,$$

where $(P-\varepsilon b)\,a^\varepsilon = (\varepsilon-\varepsilon)\,b = 0$, and consequently $(P-b\delta)\,a^\varepsilon T = \{P-(\delta-\varepsilon)\,b\}\,T$; viz. for the operation upon T we regard $P-\delta b$ as standing for $P-(\delta-\varepsilon)\,b$. As regards Q we have $(Q-2\omega b)\,a^\varepsilon T = (Q-2\omega b)\,T$; viz. the degree of T does not here present itself.

80. We may write

$$(2\omega P-\delta Q)\,S = S',$$

the new seminvariant S' being of the weight $\omega+1$; hence also

$$\{(2\omega+2)\,P-\delta Q\}.\{2\omega P-\delta Q\}\,S = S'',$$

where S'' is of the weight $\omega+2$; viz. we have an operator

$$\{(2\omega+2)\,P-\delta Q\}.\{2\omega P-\delta Q\},$$

which operating on a seminvariant of the deg. weight $\delta.\omega$ gives a seminvariant of the deg. weight $\delta.\omega+2$. This is

$$=(4\omega^2+4\omega)(P^2+P.P)-(2\omega+2)\delta(PQ+P.Q)-2\omega\delta\,(QP+Q.P)+\delta^2\,(Q^2+Q.Q),$$

where P^2, PQ, QP and Q^2 are the mere algebraical squares and products, while

$P.Q$ and $Q.P$ denote respectively P operating on Q and Q operating on P; and since $PQ = QP$ this is

$$=(4\omega^2+4\omega)(P^2+P.P)-(4\omega+2)\delta PQ-2(\omega+2)\delta P.Q-2\omega\delta Q.P+\delta^2(Q^2+Q.Q).$$

Recollecting that $P = b\partial_a + c\partial_b + d\partial_c + \ldots$, $Q = c\partial_b + 2d\partial_c + \ldots$, we have

$$\begin{aligned} P.P &= c\partial_a + d\partial_b + e\partial_c + \ldots, \\ P.Q &= \quad\;\; d\partial_b + 2e\partial_c + \ldots, \\ Q.P &= c\partial_a + 2d\partial_b + 3e\partial_c + \ldots, = P.P + P.Q, \\ Q.Q &= \quad\;\; 1.2d\partial_b + 2.3e\partial_c + \ldots, \end{aligned}$$

and attending to the relation just obtained $Q.P = P.P + P.Q$, we find that the operator may be written

$$\begin{aligned} &(4\omega^2 + 4\omega)\{P^2 - (\delta - 1)P.P\} \\ &- (4\omega + 2)\,\delta\,\{PQ - \omega P.P - \tfrac{1}{3}(\delta - 3)\,P.Q\} \\ &+ \delta^2\{Q^2 + Q.Q - \tfrac{1}{3}(4\omega + 2)\,P.Q\}\,; \end{aligned}$$

in fact here the terms in P^2, PQ, Q^2 are in the original form, while those in $P.P$, $P.Q$, $Q.Q$ are

$$\begin{aligned} (4\omega^2 + 4\omega)(1 - \delta)P.P + (4\omega^2 + 2\omega)\,\delta P.P - \tfrac{1}{3}(4\omega + 2)(\delta^2 - 3\delta)\,P.Q + \delta^2 Q.Q \\ + \tfrac{1}{3}(4\omega + 2)\,\delta^2 \qquad P.Q, \end{aligned}$$

which are

$$= (4\omega^2 + 4\omega - 2\omega\delta)\,P.P - (4\omega + 2)\,\delta P.Q + \delta^2 Q.Q,$$

agreeing with the original form

$$(4\omega^2 + 4\omega)\,P.P - (2\omega + 2)\,\delta P.Q - 2\omega\delta\,(P.P + P.Q) + \delta^2 Q.Q.$$

81. I find that each of the three parts is separately an operator, viz. that we have

$$\begin{aligned} &P^2 - (\delta - 1)\,P.P, \\ &PQ - \omega P.P - \tfrac{1}{3}(\delta - 3)\,P.Q, \\ &Q^2 + Q.Q - \tfrac{1}{3}(4\omega + 2)\,P.Q, \end{aligned}$$

each of them an operator, which operating on a seminvariant of deg. weight $\delta.\omega$ gives a seminvariant of deg. weight $\delta.\omega + 2$.

I verify this for the first of the three operators, say

$$\Omega = P^2 - (\delta - 1)\,P.P = P^2 + P.P - \delta\Theta,$$

if for a moment $P.P, = c\partial_a + d\partial_b + e\partial_c + \ldots$ is put $= \Theta$.

Here for a seminvariant S we have

$$\Omega S = (P^2 + P.P - \delta\Theta)\,S = P\,(PS) - \delta\Theta S.$$

Writing $S' = (aP - b\delta)\,S$, then S' is a seminvariant, degree $= \delta + 1$, and then if $S'' = (aP - b\,(\delta + 1))\,S'$, S'' is a seminvariant, degree $= \delta + 2$. We have $PS = a^{-1}\,(S' + b\delta S)$, and thence

$$\Omega S = Pa^{-1}\,(S' + b\delta S) - \delta\Theta S, = -\,b\,(S' + b\delta S) + P\,(S' + b\delta S) - \delta\Theta S.$$

Here

$$P(S' + b\delta S) = PS' + c\delta S + b\delta PS, = S'' + b\,(\delta + 1)\,S' + c\delta S + b\delta\,(S' + b\delta S),$$

and hence

$$\Omega S = S'' + 2b\delta S' + \{c\delta + b^2\,(\delta^2 - \delta)\}\,S - \delta\Theta S.$$

This will be a seminvariant if $\Delta.\Omega S = 0$; we have

$$\begin{aligned}\Delta.\Omega S = \Delta S'' + 2b\delta\Delta S' + \{c\delta &+ b^2\,(\delta^2 - \delta)\}\,\Delta S - \delta\,(\Delta\Theta + \Delta.\Theta)\,S \\ + 2\delta S' \quad + \{2b\delta &+ 2b\,(\delta^2 - \delta)\}\,S,\end{aligned}$$

or omitting the terms in $\Delta S''$, $\Delta S'$, ΔS which respectively vanish, this is

$$= 2\delta S' + 2b\delta^2 S - \delta\,(\Delta\Theta + \Delta.\Theta)\,S.$$

But since $PS = S' + b\delta S$, and from $\Delta S = 0$ we deduce $0 = (\Theta\Delta + \Theta.\Delta)\,S$, the equation becomes

$$\Delta.\Omega S = 2\delta PS - \delta(\Delta.\Theta - \Theta.\Delta)\,S,$$

and from $\Delta = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots,\ \Theta = c\partial_a + d\partial_b + e\partial_c + \ldots$ we have

$$\begin{aligned}\Delta.\Theta &= 2b\partial_a + 3c\partial_b + 4d\partial_c + \ldots \\ \Theta.\Delta &= c\partial_b + 2d\partial_c + \ldots,\end{aligned}$$

and thence

$$\Delta.\Theta - \Theta.\Delta = 2b\partial_a + 2c\partial_b + 2d\partial_c + \ldots, = 2P,$$

and we have thus the required equation $\Delta.\Omega S = 0$.

82. If instead of P, Θ, we write B, C, so that

$$\begin{aligned} B &= b\partial_a + c\partial_b + d\partial_c + \ldots \\ C = B.B &= c\partial_a + d\partial_b + e\partial_c + \ldots, \text{ and put further} \\ D = B.C &= d\partial_a + e\partial_b + f\partial_c + \ldots \\ E = B.D &= e\partial_a + f\partial_b + g\partial_c + \ldots,\end{aligned}$$

then the foregoing operator is $B^2-(\delta-1)\,C$, or reversing the sign, say it is $(\delta-1)\,C-B^2$, which is the first of a series of operators

$$\begin{array}{l}(\delta-1)\,C-B^2,\\ (\delta-1)(\delta-2)\,D-3\,(\delta-2)\,BC \qquad +2B^3,\\ (\delta-1)(\delta-2)(\delta-3)\,E-4(\delta-2)(\delta-3)\,BD+6\,(\delta-3)\,B^2C-3B^4,\\ \vdots\end{array}$$

which are of the deg. weights 0.2, 0.3, 0.4, etc., respectively, viz. operating upon a seminvariant of deg. weight $\delta.\omega$ they leave the degree unaltered, but increase the weight by 2, 3, 4, ... respectively.

It is to be observed that B^2, BC, B^3, etc., denote the mere algebraical powers and products of the symbols B, C, D, etc., without any operation of one symbol on another.

As a simple illustration take $(C-B^2)(ac-b^2)$, here:

$$\begin{array}{rcl} & C\,(ac-b^2)= & e-2bd+\ c^2\\ -B^2=-(2bd\partial_a\partial_c+c^2\partial_b^2)\,(\quad \text{“}\quad) & & \underline{\quad -2bd+2c^2}\\ & \text{Value is} & e-4bd+3c^2\end{array}$$

and similarly for $(C-B^2)\,(ae-4bd+3c^2)$, here:

$$\begin{array}{rcl} & C\,(ae-4bd+3c^2)= & g-4bf+(6+1)ce-4d^2\\ -B^2=-(2bf\partial_a\partial_e+2ce\partial_b\partial_d+d^2\partial_c^2)\,(\quad\text{“}\quad) & & \underline{-2bf\qquad +8ce-6d^2}\\ & \text{Value is} & g-6bf\qquad +15ce-10d^2\end{array}$$

A direct proof may of course be obtained for any one of the foregoing operators; viz. calling it Ω, it may be shown that $\Delta\Omega S$ is $=0$. I have not considered the like question of the derivation of series of operators from the other two forms

$$PQ-\omega P.P-\tfrac{1}{3}(\delta-3)P.Q \text{ and } Q^2+Q.Q-\tfrac{1}{3}(4\omega+2)P.Q \text{ respectively.}$$

83. I do not wish in the present paper to go into the theory of covariants, but it is nevertheless proper to point out the connexion which exists between the covariant theory of derivation and the operators P and Q.

Consider a quantic $(a, b, c, \ldots . a' = a_{\sigma'}\between x, y)^{\sigma'}$; any covariant hereof is $(A, B, C\ldots . \between x, y)^{\mu}$ where A is a seminvariant say of degree δ and weight,

$\omega=\frac{1}{2}(\sigma'\delta-\mu)$, or $\mu=\sigma'\delta-2\omega$, reduced to zero by the operation $\Delta=a\partial_b+2b\partial_c+\ldots.+\sigma' b'\partial_{a'}$: and if we write

$$\phi_{\sigma'}=\sigma' b\partial_a+(\sigma'-1)c\partial_b\ldots.+a'\partial_b\,,$$

then
$$B=\phi_{\sigma'}A,\ C=\tfrac{1}{2}\phi_{\sigma'}B,\ D=\tfrac{1}{3}\phi_{\sigma'}C,\ \ldots.$$

The derivative $(f,\ F)$ is $=\partial_x f\,.\,\partial_y F-\partial_y f\,.\,\partial_x F$

$$\begin{aligned}&=(a,\ b,\ \ldots.\between x,\ y)^{\sigma'-1}Bx^{\mu-1}+\ldots.\\ &-(b,\ c,\ \ldots.\between x,\ y)^{\sigma'-1}\mu Ax^{\mu-1}+\ldots.\\ &=(aB-\mu bA,\ \ldots.\between x,\ y)^{\sigma'+\mu-2}\end{aligned}$$

that is A being a seminvariant, we have $aB-\mu bA$ a seminvariant, or say

$$(\phi_{\sigma'}-\mu' b)A=\text{sem.}\ \mu'=\sigma'\delta-2\omega\,,$$

and similarly

$$(\phi_\sigma-\mu b)\,A=\text{sem.}\ \mu=\sigma\delta-2\omega\,.$$

Hence

$$\{\phi_\sigma-\phi_{\sigma'}-(\mu-\mu')b\}A,\ \text{and}\ \{\sigma'\phi_\sigma-\sigma'\phi_\sigma-(\sigma'\mu-\sigma\mu')b\}A$$

are each of them a seminvariant: but

$$\begin{aligned}\phi_\sigma&=\sigma b\partial_a+(\sigma-1)c\partial_b+\ldots.,\\ \phi_{\sigma'}&=\sigma' b\partial_a+(\sigma'-1)c\partial+\ldots.,\end{aligned}$$

$$\phi_\sigma-\phi_{\sigma'}=(\sigma-\sigma')(b\partial_a+c\partial_b+\ldots.)=(\sigma-\sigma')P,\ \mu-\mu'=(\sigma-\sigma')\delta,$$

and first form, omitting factor $\sigma-\sigma'$, is $=(P-\delta b)A$: similarly $\sigma'\phi_\sigma-\sigma\phi_{\sigma'}=(\sigma-\sigma')(c\partial_b+2d\partial_c+\ldots.)=(\sigma-\sigma')Q$ and $\sigma'\mu-\sigma\mu'=(\sigma-\sigma')2\omega$, and second form is $=(Q-2\omega b)A$.

We thus see that the operators $P-\delta b$ and $Q-2\omega b$ upon a seminvariant A depend on the derivation of f upon a covariant which has A for its leading coefficient: the order of f is arbitrary and we have thus two distinct forms.

84. As an illustration consider the quantics $(1,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5$, and $(1,\ b,\ c,\ d,\ e,\ f,\ g\between x,\ y)^6$: each of these has a covariant the leading coefficient of which is $A=f-5be+2cd+8b^2d-6bc^2$, viz. these are

$$\left(\begin{array}{r|r|l} f+1 & bf+5 & \ldots. \\ be-5 & ce-16 & \\ cd+2 & d^2+6 & \\ b^2d+8 & b^2e-9 & \\ bc^2-6 & bcd+38 & \\ & c^3-24 & \end{array}\between x,\ y\right)^5 \quad \text{and} \quad \left(\begin{array}{r|r|l} f+1 & g+1 & \ldots. \\ be-5 & bf+2 & \\ cd+2 & ce-19 & \\ b^2d+8 & d^2+8 & \\ bc^2-6 & b^2e-6 & \\ & bcd+44 & \\ & c^3-30 & \end{array}\between x,\ y\right)^8$$

$$\pm 11 \quad \pm 49 \qquad\qquad \pm 11 \quad \pm 55$$

and we find without difficulty

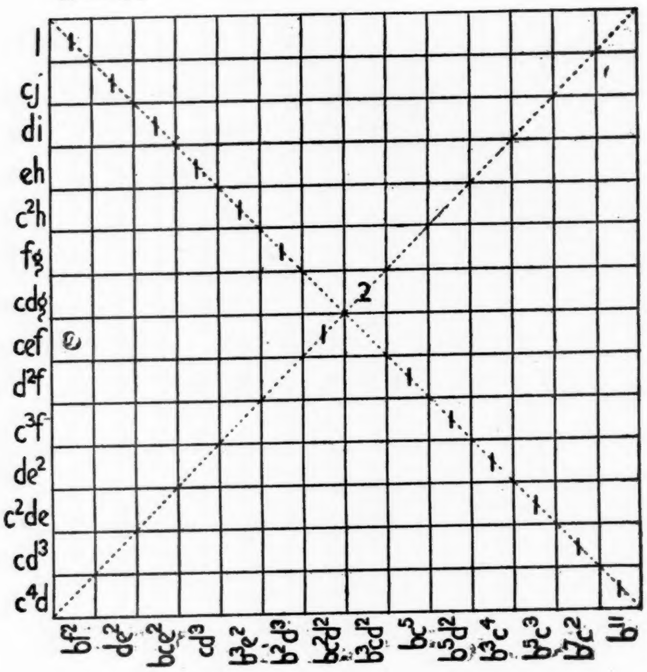
		$(g \infty d^2)$	$(ce \infty c^3)$	$(d^2 \infty b^2c^2)$
(f_1, F_1)	$=$		-16	-10
(f_2, F_2)	$=$	1	-34	-16
$(P - 3b)A$	$=$	1	-18	-6
$(Q - 10b)A$	$=$	5	-74	-20

and thence

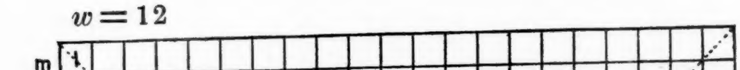
$$(P - 3b)A = (f_2, F_2) - (f_1, F_1),$$
$$(Q - 10b)A = 5(f_2, F_2) - 6(f_1, F_1),$$

viz. we thus have $P - 3b$, and $Q - 10b$ upon $f \infty bc^2$ each given as a linear function of the derivatives (f_1, F_1) and (f_2, F_2) where f_1, f_2 are the quintic and the sextic function, and F_1, F_2 are like covariants of these functions respectively.

$w = 11$


l
cj
di
eh
c²h
fg
cdg
cef
d²f
c³f
de²
c²de
cd³
c⁴d

$w = 12$


m
ck
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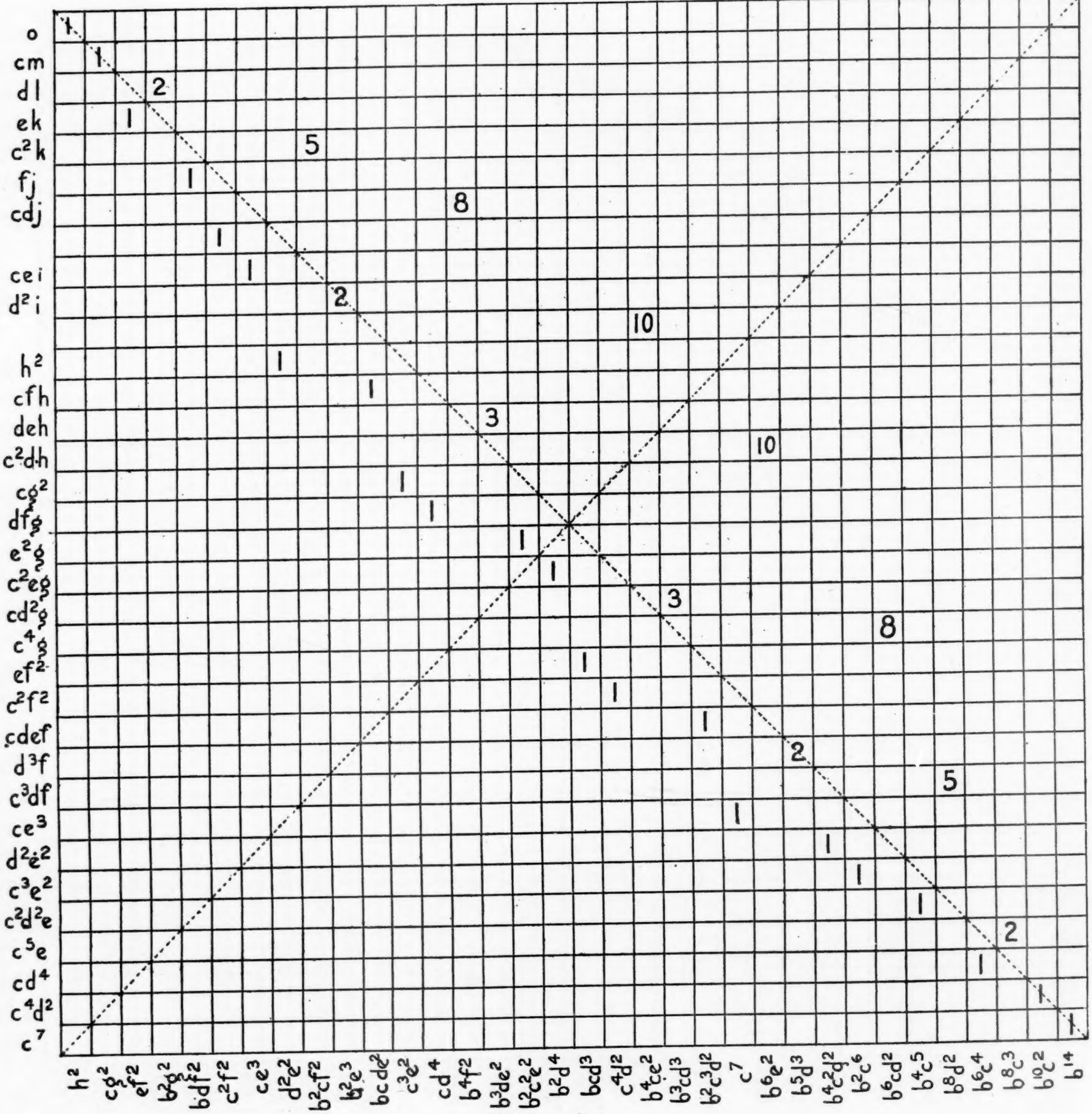
ei
c²i
fh
cdh
g²
ceg
d²g
c³g
cf²
def
c²df
e³
c²e²
cd²e
c⁴e
d⁴
c³d²
c⁶

$w = 13$

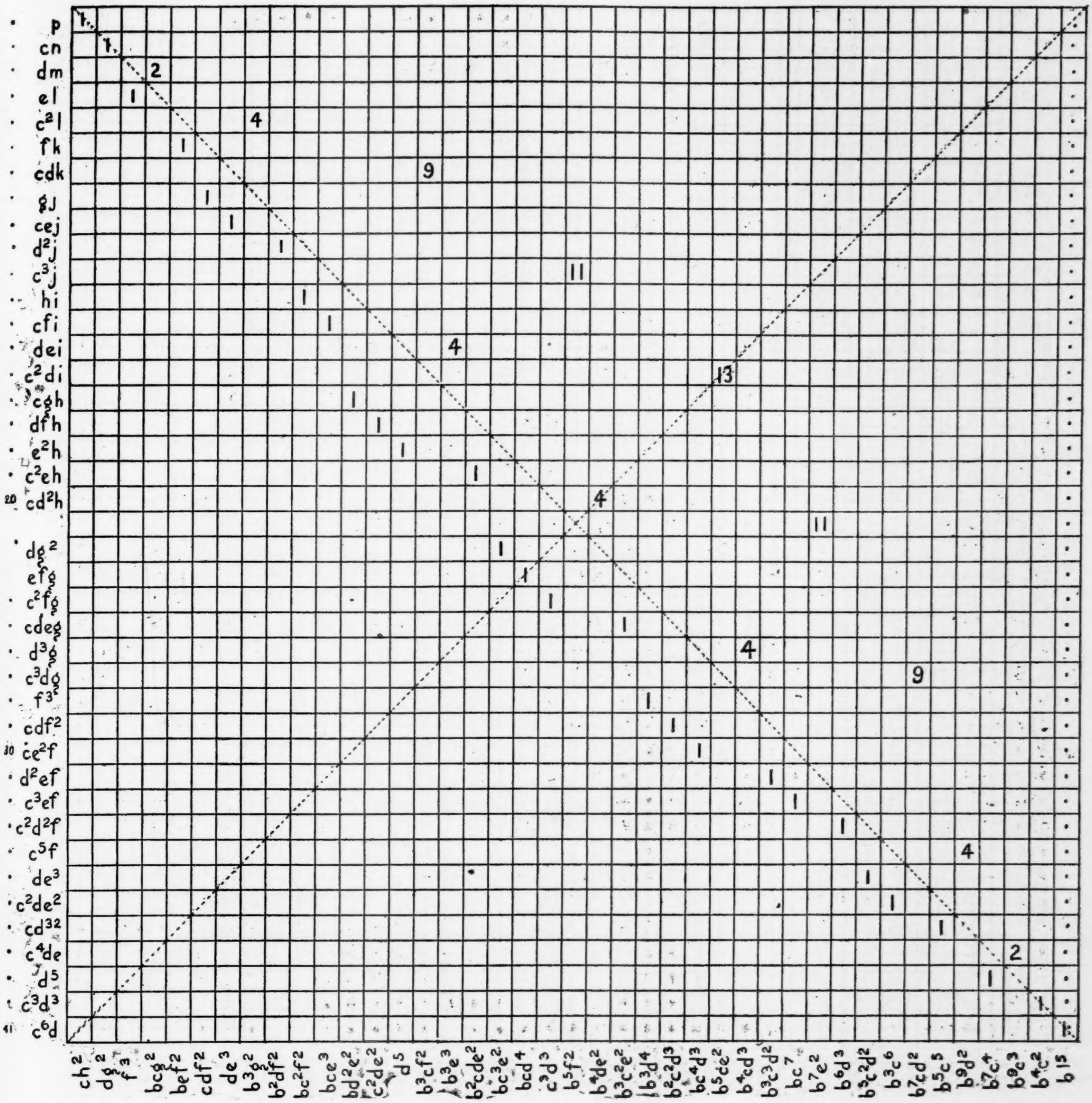
	bg^2	df^2	bcf^2	be^3	cde^2	b^3f^2	b^2de^2	bc^2e^2	bd^4	c^2d^3	b^3ce^2	b^2cd^3	bc^3d^2	b^5e^2	b^4d^3	$b^3c^2d^2$	bc^6	b^5cd^2	b^3c^5	b^7d^2	b^5c^4	b^7c^3	b^9c^2	b^{13}
n	1																							
cl		1																						
dk			1																					
ej				1																				
c^2j						2																		
fi					1																			
cdi											5													
gh							1																	
ceh								1																
d^2h									1															
c^3h														4										
cfg										1														
deg												1												
c^2dg																		5						
df^2													1											
e^2f															1									
c^2ef																1								
cd^2f																	1							
c^4f																				2				
cde^2																			1					
d^3e																					1			
c^3de																						1		
c^2d^3																							1	
c^5d																								1

$w = 14$



o
cm
dl
ek
c^2k
fj
cdj
cei
d^2i
h^2
cfh
deh
c^2dh
cg^2
dfg
e^2g
c^2eg
cd^2g
c^4g
ef^2
c^2f^2
cdef
d^3f
c^3df
ce^3
d^2e^2
c^3e^2
c^2d^2e
c^5e
cd^4
c^4d^2
c^7
h^2
cg^2
ef^2
b^2g^2
bdf^2
c^2f^2
ce^3
d^2e^2
b^2cf^2
b^2e^3
$bcde^2$
c^3e^2
cd^4
b^4f^2
b^3de^2
$b^2c^2e^2$
b^2d^4
bcd^3
c^4d^2
b^4ce^2
b^3cd^3
$b^2c^3d^2$
c^7
b^6e^2
b^5d^3
$b^4c^2d^2$
b^2c^6
b^6cd^2
b^4c^5
b^8d^2
b^6c^4
b^8c^3
$b^{10}c^2$
b^{14}

$w = 15$


p
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c^2l
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gj
cej
d^2j
c^3j
hi
cfi
dei
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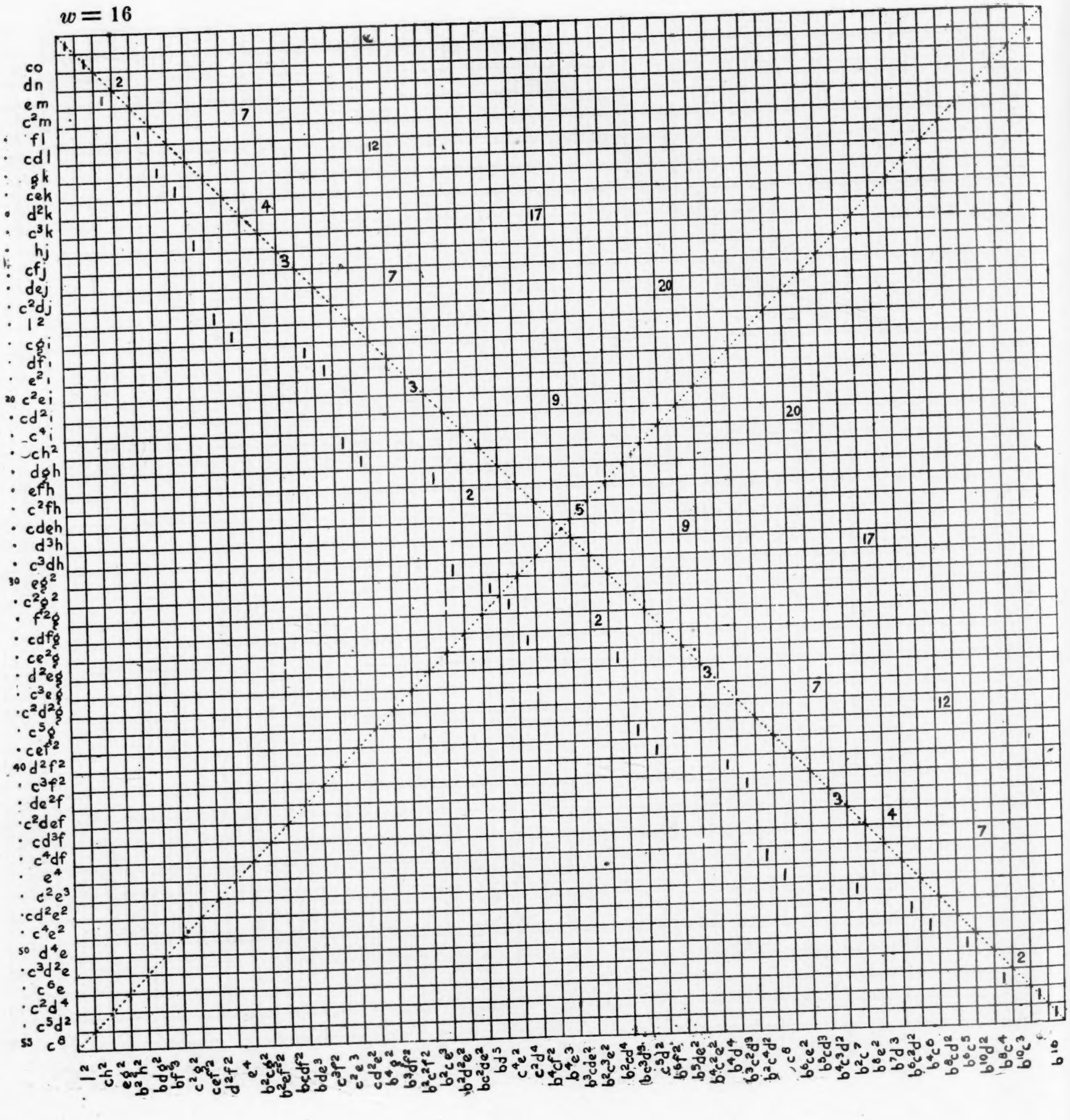
cgh
dfh
e^2h
c^2eh
cd^2h
dg^2
efg
c^2fg
cdeg
d^3g
c^3dg
f^3
cdf^2
ce^2f
d^2ef
c^3ef
c^2d^2f
c^5f
de^3
c^2de^2
cd^{32}
c^4de
d^5
c^3d^3
c^6d

$w = 16$


co
dn
em
c^2m
fl
cdl
gk
cek
d^2k
c^3k
hj
cfj
dej
c^2dj
i^2
cgi
dfi
e^2i
c^2ei
cd^2i
c^4i
ch^2
dgh
efh
c^2fh
cdeh
d^3h
c^3dh
eg^2
c^2g^2
f^2g
cdfg
ce^2g
d^2eg
c^3eg
c^2d^2g
c^5g
cef^2
d^2f^2
c^3f^2
de^2f
c^2def
cd^3f
c^4df
e^4
c^2e^3
cd^2e^2
c^4e^2
d^4e
c^3d^2e
c^6e
c^2d^4
c^5d^2
c^8

Tables of Pure Reciprocants to the Weight 8.

BY PROF. CAYLEY.

In the tabulation of Pure Reciprocants it is convenient to write $a = 1$; we thus have for all the reciprocants of a given weight a single column of literal terms which (as in the Seminvariant Tables) I arrange in alphabetical order AO, and the several reciprocants have then each of them its own column of numerical coefficients: the form of the table is thus similar to that of the seminvariant table, the only difference being that for reciprocants the final terms are not in general power-enders: as in the seminvariant table, the columns of the table are arranged *inter se* with their final terms in AO. As remarked in my paper, "Corrected Seminvariant Tables for the Weights 11 and 12," A. M. J., t. XIV (1892), pp. 195–200, it is not in every case the top term of a column which should be regarded as the initial term; but to the extent 8 to which the reciprocant tables are here carried this remark has no application.

I recall that the notation is the modified one employed by Halphen, and by Sylvester in his 12th and subsequent lectures, viz. $a, b, c, d, \ldots$ denote $\frac{1}{2}\frac{d^2y}{dx^2}, \frac{1}{6}\frac{d^3y}{dx^3}, \frac{1}{24}\frac{d^4y}{dx^4}, \frac{1}{120}\frac{d^5y}{dx^5}, \ldots$ respectively. As already noticed, a is put $= 1$, but it is to be in the several terms restored in the proper powers so as to obtain for the reciprocant a homogeneous expression of a degree equal to the original degree of the final term; thus $d - 3bc + 2b^3$ is to be read as standing for $a^2d - 3abc + 2b^3$.

The ultimate verification of the expression for a pure reciprocant consists (as is known) in its annihilation by the operator

$$V = 2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d + (7ad + 7bc)\partial_e + (8ae + 8bd + 4c^2)\partial_f + \text{etc.},$$

or say

$$V = \;\; 2\partial_b + \;\; 5b\partial_c + \;\; (6c + 3b^2)\partial_d + \;\; (7d + 7bc)\partial_e + \;\; (8e + 8bd + 4c^2)\partial_f + \text{etc.};$$

thus for the reciprocant $50e - 175bd + 28c^2 + 105b^2c$, the result obtained is

$$2(-175d + 210bc) + 5b(56c + 105b^2) + (6c + 3b^2)(-175b) + (7d + 7bc)(50),$$

or collecting, this is

d	-350			$+350$	± 350
bc	$+420$	$+280$	-1050	$+350$	± 1050
b^3		$+525$	-525		± 525

$= 0$, as it should be.

The tables are

c	$+4$
b^2	-5
	$+4$
	-5

d	$+1$
bc	-3
b^3	$+2$
	± 3

e	$+50$	
bd	-175	
c^2	$+28$	$+16$
b^2c	$+105$	-40
b^4		$+25$
	$+183$	$+41$
	-175	-40

f	$+10$	
be	-40	
cd	-12	$+4$
b^2d	$+65$	-5
bc^2	$+16$	-12
b^3c	-39	$+23$
b^5		-10
	± 91	± 27

g	$+14$			
bf	-63			
ce	-1350	$+800$		
d^2	$+1470$	-875	$+125$	
b^2e	$+1782$	-1000		
bcd	-4158	$+2450$	-750	
c^3	$+2130$	-1344	$+256$	$+64$
b^3d			$+500$	
b^2c^2		$+35$	$+165$	-240
b^4c			-300	$+300$
b^6				-125
	$+5576$	$+3250$	± 1018	$+364$
	-5508	-3254		-365

h	$+7$			
bg	-35			
cf	-539	$+560$		
de	$+605$	-650	$+50$	
b^2f	$+735$	-700		
bce	$+306$	-290	-150	
bd^2	-2135	$+2275$	-175	
c^2d	$+1001$	-1036	$+28$	$+16$
b^3e	-1485	$+1500$	$+100$	
b^2cd	$+3465$	-3710	$+630$	-40
bc^3	-1295	$+1988$	-84	-48
b^4d			-350	$+25$
b^3c^2		$+63$	-259	$+152$
b^5c			$+210$	-155
b^7				$+50$
	$+6119$	± 6386	± 1018	± 243
	-5489			

i	+ 420						
bh	− 2310						
cg	− 32704	+ 1176					
df	+ 57750	− 8085	+ 20433				
e^2	− 20460	+ 7040	− 21542	+ 625			
b^2g	+ 45500	− 1470					
bcf	− 28392	+ 18963	− 61299				
bde	− 90900	− 16940	+ 69062	− 4375			
c^2e	+ 103740	− 27160	+ 80248	− 49700	+ 3200		
cd^2	− 38320	+ 26460	− 85554	+ 55125	− 3500	+ 500	
b^3f	− 69615	− 9555	+ 40866				
b^2ce	+ 83538	+ 28098	− 106218	+ 128625	− 8000		
b^2d^2	+ 92820	+ 12740	− 54782	− 61250	+ 4375	− 625	
bc^2d	− 102102	− 52822	+ 191590	− 156800	+ 9800	− 3000	
c^4		+ 21560	− 73304	+ 84868	− 5376	+ 1024	+ 256
b^4e			− 378	− 78750	+ 5000		
b^3cd			+ 1176	+ 183750	− 12250	+ 5750	
b^2c^3				− 102165	+ 6580	− 620	− 1280
b^5d						− 2500	
b^4c^2					+ 175	− 2025	+ 2400
b^6c						+ 1500	− 2000
b^8							+ 625
	+ 383768	+ 116037	+ 403375	+ 452993	+ 29130	+ 8774	+ 3281
	− 384803	− 116032	− 403077	− 453040	− 29126	− 8750	− 3280

I remark that in the last of these tables the first column, say $i \infty bc^2d$, which ends in bc^2d, is a more simple form than Sylvester's P_8, $= i \infty c^4$, (A. M. J., t. IX, p. 35) which ends in c^4; P_8 is in fact a linear combination, first col. + 6 second col. of the first and second columns of the table: the second column, say $cg \infty c^4$ is Sylvester's (a^2cg), t. IX, p. 124.

On the Differential Equation $\Delta u + k^2 u = 0$.

By Maxime Bôcher.

It is well known that any solution of the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0, \tag{1}$$

or as I will say for the sake of brevity any u-function, yields when multiplied by the factor $e^{\pm kz}$ a Newtonian potential function, that is a solution of the equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{2}$$

It has not, however, as far as I have been able to ascertain, been noticed that this fact can be made use of to deduce a considerable number of the fundamental properties of u-functions of two variables from well known properties of the Newtonian potential function. It is my purpose in the following paper to show in some detail how this can be done.

It is true that the method here suggested has only a limited range of application, and that from the point of view of the purist (I use the term in no invidious sense), the processes employed by H. Weber* and Pockels,† which consist in generalizing the methods and formulæ of the theory of the *two* dimensional potential, are vastly to be preferred. The course pursued in the present article, however, has the advantage of arriving at many of the most important results with very little labor when once the properties of the Newtonian potential function are premised.‡

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* "Ueber die Integration der partiellen Differentialgleichung: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$." Mathematische Annalen, Vol. I.

† "Ueber die partielle Differentialgleichung $\Delta u + k^2 u = 0$." Teubner, Leipzig, 1891.

‡ In the present paper I shall confine my attention to u-functions with two independent variables. u-functions with n independent variables, i. e. solutions of the equation

$$\sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2} + k^2 u = 0,$$

may, however, be treated in precisely the same way. Their properties would then be made to depend upon the properties of the potential in space of $n+1$ dimensions.

Instead of dealing with either of the Newtonian potential functions $e^{\pm kz} \,.\, u(x, y)$, it will usually be more convenient to take as our potential a combination of the two, namely

$$\cosh kz \,.\, u(x, y).$$

One advantage of this modification is that when k has the pure imaginary value $i\varkappa$ this potential is still real, having the form

$$\cos \varkappa z \,.\, u(x, y).$$

The u-functions whose k is real have in many respects different properties from those whose k is pure imaginary. One of the most fundamental of these differences is the following:

*A u-function with pure imaginary k cannot vanish at all points of the boundary of a region which lies in a portion of the x, y plane where the function is finite, continuous and single valued. When, however, k is real, u-functions do exist which vanish along the boundaries of such regions.**

In order to establish the first part of this proposition we have merely to notice that any vanishing line of a u-function in the x, y plane gives a cylindrical surface whose elements are parallel to the axis of z, at every point of which the potential $\cos \varkappa z \,.\, u(x, y)$ vanishes. This potential, however, vanishes also on an infinite number of planes parallel to the plane x, y. Accordingly if there did exist a region of the nature above described on whose boundary the u-function vanished, the potential function would vanish on the boundary of an infinite number of finite solids cut out by the cylindrical surface just mentioned and by planes perpendicular to its elements. This, however, we know from the theory of the potential to be impossible.

On the other hand when k is real the above reasoning does not apply; for the potential $\cosh kz \,.\, u(x, y)$ vanishes only on the cylindrical surfaces erected on the vanishing curves of the u-function, and space is not cut up into finite regions on whose boundaries the potential vanishes. That when k is real we actually do have such exceptional regions† in the x, y plane on whose boundaries the u-function vanishes is seen by numerous familiar examples.

Another proposition which we can obtain with ease is that, whether k be real or imaginary, *no point at which a u-function vanishes can be isolated, but every*

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* Pockels, p. 189; pp. 33–186.

† "Ausgezeichnete Bereiche"; cf. Pockels, p. 222.

*such point must lie on a curve at all of whose points the u-function vanishes.** For if such an isolated point existed, the potential $\cosh kz . u(x, y)$ would vanish along an isolated line, whereas we know that it must vanish along a whole surface.

Now it is equally true that every equipotential surface $V =$ constant (not merely those where $V = 0$) must be a real surface and not an isolated point or line. We cannot, however, conclude from this that the points where a u-function has a given value must completely fill a line, for the surfaces along which $\cosh kz . u(x, y)$ is constant are not cylindrical surfaces except when this constant has the value zero. The sections of these equipotential surfaces with the plane x, y give the lines along which the u-function has constant values. It is, however, perfectly conceivable that some of these equipotential surfaces should cut the plane x, y in one or more isolated points. That this really does occur is seen most readily by considering the u-function $J_0(k\sqrt{x^2 + y^2})$ where J_0 denotes a Bessels function of the first kind of order zero.

It should be noticed that isolated points of this kind, at which of course the u-function reaches a maximum or a minimum, may occur not merely when k is real but also when it is pure imaginary. There is, however, an essential distinction between the two cases which is not explicitly stated by Weber or Pockels,† and which may be stated as follows:

Within a region where a u-function is finite, continuous and single valued there may exist points where it reaches a maximum or a minimum. If, however, k is real, the ABSOLUTE VALUE *of u must have a maximum at these points; if k is pure imaginary, a minimum.*

To prove this proposition we may assume that the u-function has a positive value at the point in question, for if it had not we could get a u-function which would be positive there by multiplying by -1. Then we merely have to show that if k is real, u cannot be a minimum, and that if k is pure imaginary it cannot be a maximum. This, however, follows at once from the fact that k being real and u a positive minimum, the potential $\cosh kz . u(x, y)$ would itself be a minimum at the point in question; while if $k = i\varkappa$ were pure imaginary and u a positive maximum, the potential $\cos \varkappa z . u(x, y)$ would be a maximum.

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* Pockels, p. 218.

† It is, however, implicitly contained in the last paragraph on page 10 of Weber's article quoted above. I find, however, no indication of the possibility of extending the proposition to the case where u is a maximum or a minimum along a curve.

For the same reasons the above proposition holds true when for maximum or minimum *points* of the u-function we substitute maximum or minimum *lines*. The value of the u-function on such lines cannot, moreover, be zero any more than it can at maximum or minimum points. This also follows easily from the theory of the Newtonian potential function.*

Another proposition, whose truth we see at a glance, is the one established by Pockels (p. 226), which says that *two vanishing lines of a u-function cut each other, if at all, orthogonally, except when n vanishing lines pass through the same point, in which case they make equal angles* $\frac{\pi}{n}$ *with one another.* This follows from the corresponding proposition concerning the Newtonian potential, since the vanishing surfaces of the potential $\cosh kz . u(x, y)$ are merely the cylinders erected on the vanishing lines of the u-function as base. Here again, however, we can draw no inference concerning the lines along which a u-function has a constant value other than zero.

The theory of the Newtonian potential tells us that a potential V can be developed about any non-singular point into a series proceeding according to ascending spherical harmonics, this development holding within a sphere described about the point as centre and passing through the nearest singular point of the potential function. If we apply this proposition to the special class of potentials we are here interested in we get the following development:

$$\cosh kz . u(x, y) = \Phi_0 + r\Phi_1 + r^2\Phi_2 + \ldots .$$

where the surface spherical harmonic Φ_n may, by using polar coordinates, be written in the form:

$$\Phi_n =$$
$$a_{n,0}P_n(\cos\vartheta) + a_{n,1}P_n^1(\cos\vartheta).\cos\phi + a_{n,2}P_n^2(\cos\vartheta).\cos 2\phi + \ldots + a_{n,n}P_n^n(\cos\vartheta).\cos n\phi$$
$$+ b_{n,1}P_n^1(\cos\vartheta).\sin\phi + b_{n,2}P_n^2(\cos\vartheta).\sin 2\phi + \ldots + b_{n,n}P_n^n(\cos\vartheta).\sin n\phi .$$

We will suppose the origin of the system of polar coordinates r, ϑ, ϕ to be in the x, y plane, and the axis of the system to be perpendicular to this plane. If now in the development we set $z = 0$ (i.e. $\vartheta = \frac{\pi}{2}$) we get:

$$u = X_0 + rX_1 + r^2X_2 + \ldots . \tag{3}$$

*** It may be interesting to note that at infinity just the opposite of this is true, inasmuch as there a u-function may have zero as a maximum or minimum value, and that this is the only case where a u-function can fail to have a (real) singularity at infinity.**

where

$$X_n = A_{n,0} + A_{n,1}\cos\phi + A_{n,2}\cos 2\phi + \ldots + A_{n,n}\cos n\phi$$
$$+ B_{n,1}\sin\phi + B_{n,2}\sin 2\phi + \ldots + B_{n,n}\sin n\phi.*$$

We thus get the proposition:

In the neighborhood of any non-singular point a u-function may be developed in a series of the form (3), *and this series will converge within a circle described about the point in question as centre and passing through the nearest singular point of the u-function.*

If we compare the development (3) with the development

$$u = \sum_{n=0}^{n=\infty} J_n(kr) . (A_n \cos n\phi + B_n \sin n\phi) \tag{4}$$

(cf. Pockels, p. 226) we see that each term of (4) is a u-function while the terms of (3) are not. This makes (4), as Pockels says, the natural generalization of the development

$$V(x, y) = \sum_{n=0}^{n=\infty} r^n . (A_n \cos n\phi + B_n \sin n\phi) \tag{5}$$

of a two dimensional potential. The region of convergency of the series (4) has not, however, been investigated, while, as above shown, the series (3) has a circle of convergency exactly like that of (5).

A number of other propositions in the theory of the Newtonian potential yield in the same way simple properties of the two dimensional u-function.†

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* These functions X_n are considerably simplified, owing to the fact that all of their coefficients $A_{n,m}$ and $B_{n,m}$ vanish in which $n-m$ is an odd number, since the same is true of $P_n^m(0)$ which enters as a factor into them.

† On the other hand, many simple properties of the Newtonian potential yield propositions concerning the u-function which are so complicated as to be of comparatively little interest. Thus the proposition that the average value of a potential on the surface of a sphere is equal to the value of the potential at the centre of the sphere gives the following:

If r_1 is the radius of a circle which lies in a region of the x, y plane where a u-function is finite, continuous and single valued, and if r denotes the (variable) distance from the centre of this circle, then the average value of

$$\frac{r_1 \cosh(k\sqrt{r_1^2 - r^2})}{2\sqrt{r_1^2 - r^2}} . u(x, y).$$

within the circle is equal to the value of u at the centre of the circle.

By comparing this complicated proposition with the simple one to which it bears a certain resemblance (equation 66, Pockels, p. 217) we get the following rather simple definite integral formula, which, of course, has nothing to do with the theory of u-functions:

$$x = \int_0^x \cos z . J_0(\sqrt{z^2 - x^2}) dz.$$

One of the more important propositions which can be proved at once in this way will be found on page 212 of Pockels' book.

Still another application of our method should be mentioned, although the limits of the present article forbid a detailed discussion of it.

An extended class of problems in the theory of the potential may be solved by the method of development in series which proceed according to trigonometric functions, spherical harmonics, or other similar functions. The same is true in the theory of the u-functions.* In the theory of the potential a simple theory has been found by Klein† connecting the great number of isolated problems which had previously been solved by development in series. The same theory may be at once applied to u-functions (of any number of variables) by means of the method of the present article. I expect on another occasion to return to this question.

In conclusion we must note that the method of the present article is applicable not merely to the equation $\Delta u + k^2 u = 0$, but also to a number of other partial differential equations, chief among these being the equation for surface spherical harmonics. We could get by means of our method a series of propositions concerning surface spherical harmonics closely resembling those deduced in the present article for u-functions; and in one respect we should be even freer in the application of our method, as we should not be hampered by the presence of an essential singular point, such as we have at infinity in the case of the potential $\cosh kz . u(x, y)$. The method employed by Prof. Klein in his lectures to prove that complete surface spherical harmonics must be of integral order‡ comes under the method of the present article.

HARVARD UNIVERSITY, *August* 2, 1892.

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* See Pockels, pp. 326–335.

† See his note in the Göttinger Nachrichten for March, 1890; and also my essay: "Ueber die Reihenentwickelungen der Potentialtheorie," Göttingen, 1891.

‡ See Pockels, p. 106.

Geometrical Illustrations of Some Theorems in Number.

By Ellery W. Davis.

The Cartesian coordinates of a point in a plane being denoted by x and y, consider a change to a system whose coordinates, ξ, η, are given by

$$\xi = x \pm y, \quad \eta = y.$$

The transformation preserves all areas and collineations unchanged, together with the order of all directions about any and all points and of all positions along any and all lines.

Let x and y be integers; then also are ξ and η. Moreover, ξ and η are or are not relatively prime according as x and y are.

All that we have said would likewise be true had the transformation been

$$\xi = x, \quad \eta = y \pm x.$$

Furthermore, by a continued repetition of these transformations we can get from any point whose coordinates are relatively prime integers to all other points whose coordinates are relatively prime integers.

For suppose both coordinates positive. Then, by always retaining the smaller coordinate and replacing the larger by the difference of the two, we get at last the point (1, 1); while, by merely reversing the operations that would carry us from a point (x, y) to a point (1, 1), we get from (1, 1) to (x, y). Obvious enough is the modification of this process necessary when one or both coordinates are negative.

Imagine drawn the positive parts of the lines

$$x = 0,\ 1,\ 2,\ 3, \ldots.;\quad y = 0,\ 1,\ 2,\ 3, \ldots.$$

The intersections of these lines we call *nodes*. When a node is such that its x and y are relatively prime it is a *filled* node; otherwise the node is *open*.

Marking all the filled nodes by dots gives a diagram used by Prof. Sylvester in demonstrating certain theorems about Farey series. A modification has also been used in the calculation of factor tables.

The actual construction of the diagram is facilitated by taking account of the fact that

$$x = a, \quad y = b \qquad (a > b)$$

gives a filled node if, and only if,

$$x + y = a, \quad y = b$$

also gives one.

Checks are afforded by noticing that also

$$\begin{aligned} x + 2y &= a, & y &= b, \\ x + 3y &= a, & y &= b, \\ &\vdots & &\vdots \\ x + ky &= a, & y &= b, \end{aligned}$$

should give filled nodes when (a, b) is a filled node.

If the reader will examine on our diagram the lines cd $(x = 14)$, cef $(\pm x + y = 14)$, cgh $(\pm x + 2y = 14)$, he will find that they all pass through filled nodes for the same values of y.

A line through the origin and any filled node passes through no other filled node and through no open node between the origin and the filled node. While, since the coordinates of every open node are equimultiples of the coordinates of some filled node, every line through the origin and an open node passes through a filled node between it and the origin. That is, to an eye at the origin all of the filled nodes but none of the open nodes are visible.

Imagine the diagram continued to infinity both positively and negatively. The transformation

$$\xi = x + y, \quad \eta = y$$

then shears the diagram into itself.

In this shearing the parallel lines

$$x = 0, \ x = \pm 1, \ x = \pm 2, \ x = \pm 3, \ldots, \ x = \pm k$$

become the parallels

$$x = y, \ x = y \pm 1, \ x = y \pm 2, \ x = y \pm 3, \ldots, \ x = y \pm k.$$

Now any node on $x = \pm k$ forms with the origin and $(0, 1)$ a triangle of double-area k. Consequently any node on $x = y \pm k$ also forms with the origin and $(1, 1)$ a triangle of double-area k.

Since by a series of such transformations $(0, 1)$ can become any filled node whatever, it follows that parallel to the line joining the origin to any filled node will be pairs of lines of nodes that with the origin and the assumed filled node form triangles of double-area $1, 2, 3, \ldots, k$.

If (a, b) is the fixed assumed node we thus get all the solutions of

$$ay - bx = \pm k, \qquad (k \text{ is integral})$$

with a and b relatively prime. If x and y are also to be relatively prime we confine ourselves to filled nodes on the pairs of lines.

Similarly, taking $(a, -b)$ for the fixed node gives all the integral solutions of

$$ay + bx = \pm k.$$

If a line be drawn joining two nodes but missing the origin, there is a parallel line joining the origin to a filled node. The former line is thus gotten by a series of our transformations from a line $x = \pm k$ and so contains an infinite number of filled nodes. Otherwise there would be a number to which no other number is relatively prime.

It will be seen that the arrangement of the filled nodes on any line

$$x = \text{an integer}$$

is periodic, the period being the product of all the prime factors of the integer. Moreover, the arrangement is the same for all integers having the same prime factors, regardless of the powers to which those factors may be raised.

What has just been said remains true if for "integer" we substitute "set of integers".

Thus the arrangement of filled nodes in the strip

$$x = 1,\ 2,\ 3,\ 4$$

has the period 6; that in the strip

$$x = 1,\ 2,\ 3,\ 4,\ 5,\ 6$$

the period 30; that in the strip

$$x = 1,\ 2, \ldots,\ 25$$

the period 210.

Now the line

$$x = \text{any prime}$$

crosses filled nodes except where

$$y = \text{a multiple of that prime.}$$

Consequently a reference to the diagram shows that, having regard to periodicity, numbers greater than 3 cannot be prime unless of one of the forms

$$6n \pm 1.$$

If they are of either of these forms their least factor exceeds 3.

Similarly, numbers greater than 5 cannot be prime unless of one of the forms

$$30n \pm 1,\ 30n \pm 7,\ 30n \pm 11,\ 30n \pm 13;$$

while if of one of these forms their least factors exceed 5.

In general, if p is a prime and P the product of all the primes not greater than it, while

$$k_1,\ k_2,\ k_3,\ \ldots .$$

are the integers not greater than P whose least factor exceeds p; then numbers greater than p cannot be prime unless of one of the forms

$$Pn \pm 1,\ Pn \pm k_1,\ Pn \pm k_2,\ Pn \pm k_3,\ \ldots .,$$

while if of one of these forms their least factors exceed p.

The number of these forms is the totient of P. On our diagram it is the number of filled nodes on $x = P$ below the diagonal $x = y$.

I am indebted to Mr. W. P. Durfee for a very neat illustration of the theorem that

A number equals the sum of the totients of all its divisors.

Join the origin to all the nodes, not above $x = y$ on

$$x = N.$$

Every joining line passes through a filled node on

$$x = \text{some divisor of } N.$$

Conversely, every line from the origin through a filled node, not above $x = y$, on

$$x = \text{some divisor of } N,$$

passes through a node, not above $x = y$, on

$$x = N.$$

As all these lines pass through filled nodes, no two are the same and their number is N.

A similar illustration can be given of the theorem that

The totient of the product of two relatively prime numbers is the product of their totients.

Suppose a, b are the two relatively prime numbers and assume $a < b$.

Moreover, let p, q, r be any numbers respectively not greater than and prime to a, b, and ab.

Then (a, p) is a filled node as are also $(b, q + mb)$ and $(ab, r + nab)$ for all integral values of m and n.

We will prove that for each pair of values of p and q there is one and only one value of r such that the nodes

$$(a, p), \quad (b, q + mb), \quad (ab, r + nab)$$

are collinear.

Collinearity requires that

$$\begin{aligned} nab + r &= a(b-1)(mb + q - p)/(b - a) + p \\ &= ak(mb + q - p)/l + p \end{aligned}$$

if k/l is the fraction $(b-1)/(b-a)$ reduced to its lowest terms.

Since b is prime to $b - a$ there is, for each pair of values of p and q, just one value of m, from 1 to l inclusive, that renders $mb + q - p$ divisible by l. Thus each pair of values of p and q determines r, as an integer, uniquely, values of m outside the range giving new n's but not new r's.

The expression for $nab + r$ is evidently prime to a, and by interchanging a with b, and p with $mb + q$, we can get another expression for $nab + r$ as plainly prime to b. Thus r is prime to ab.

We will now show that different pairs of values of p and q cannot give the same r.

We have

$$mb + q = l(nab + r - p)/ak + p.$$

Here l is prime to a and consequently $r - p$ must be a multiple of a. But r being given, there is just one value of p that will render $r - p$ a multiple of a. Thus to any r there is just one p. Having thus fixed $r - p$, there is just one value of n from 1 to k inclusive that renders $nab + r - p$ divisible by k and so $mb + q$ integral. Thus q is uniquely determined.

Our two propositions established, it follows at once that the number of r's is the product of the number of p's by the number of q's.

On the diagram are represented the collineations when a, b, and ab, are respectively 3, 5, and 15.

As another case, let us take for a, b, and ab, 4, 7, and 28.

The p's are 1 and 3.

The q's are 1, 2, 3, 4, 5, and 6.

The r's are 1, 3, 5, 9, 11, 13, 17, 19, 23, 25, 27.

The collineations are

(4, 1), (7, 1), (28, 1); (4, 1), (7, 46), (28, 361);
(4, 1), (7, 16), (28, 121); (4, 1), (7, 19), (28, 145);
(4, 1), (7, 31), (28, 241); (4, 1), (7, 13), (28, 97);
(4, 3), (7, 15), (28, 99); (4, 3), (7, 18), (28, 123);
(4, 3), (7, 9), (28, 51); (4, 3), (7, 12), (28, 75);
(4, 3), (7, 24), (28, 171); (4, 3), (7, 6), (28, 27).

Since the arrangement of the filled nodes on any line $x = N$ depends only upon the prime factors of N, it follows that the fraction

$$\text{totient-of-}N/N$$

is the same for all numbers having the same prime factors.

For any prime p and its integral powers the fraction is

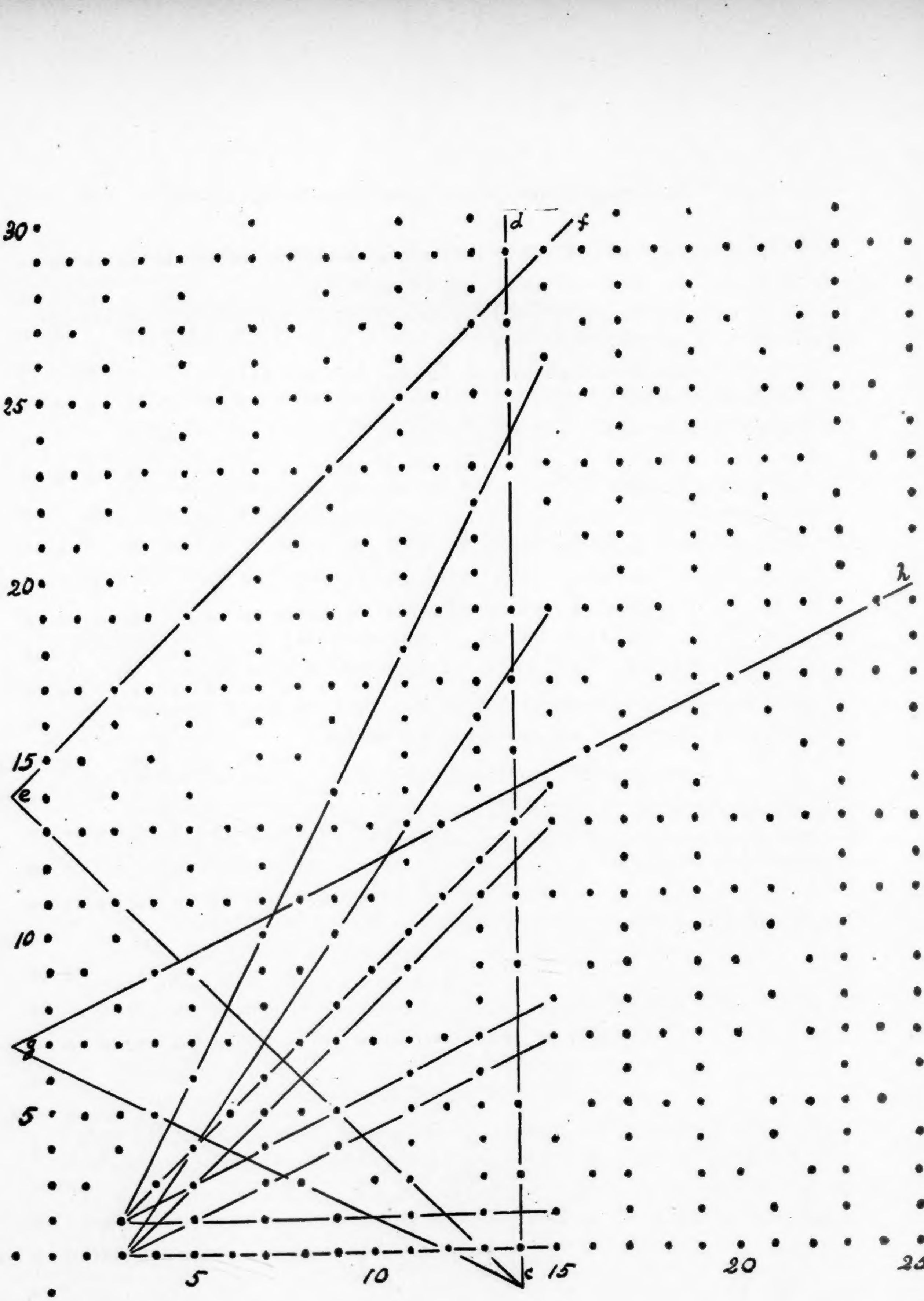
$$(p-1)/p = 1 - 1/p.$$

Consequently, for a number whose prime factors are

$$p,\ q,\ r,\ s, \ldots.$$

the value of the fraction is

$$(1 - 1/p)(1 - 1/q)(1 - 1/r)(1 - 1/s) \ldots.$$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Hyperelliptische Schnittsysteme und Zusammenordnung der algebraischen und transcendentalen Thetacharacteristiken.

Von Henry Dallas Thompson.

Einleitung.

Die hyperelliptischen Thetafunctionen können nach Massgabe der transcendentalen Characteristiken, welche Riemann in seinen Vorlesungen im Jahre 1862 gebrauchte, oder nach Massgabe der Zerspaltung der Form $f_{2p+2}(z)$ in die Factoren $\phi_{p+1-2\mu}(z) . \psi_{p+1+2\mu}(z)$ eingeteilt werden. Dass zu jeder Zerspaltung $\phi . \psi$ ein festes $\vartheta_{\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right)}$ gehört, findet man implicite angedeutet in den Artikeln, die während der sechziger Jahre erschienen sind, besonders in denen Pryms. Die systematische Zusammenordnung dieser beiden Characteristiken ist der Gegenstand dieser Abhandlung.

Durch die ganze Abhandlung hindurch werden wir versuchen, den angewandten arithmetischen Sätzen eine geometrische Form zu geben. Deshalb sind in dem in Abschnitt I gegebenen Referate ausschliesslich geometrische Ausdrücke gebraucht worden. Im Abschnitt II wird die Idee der "Geometrischen Characteristik" einer Fläche eingeführt und es werden die Figuren der verschiedenen Schnittsysteme gegeben, ganz besonders die der symmetrischen für $p = 3$ und $p = 5$, wodurch die Zusammenordnung der Thetacharacteristiken eine einfache Form annimmt. Die hier gegebenen Figuren für die sechs und dreissig Schnittsysteme für $p = 3$ sind dieselben auf die Prof. Klein in den Mathematischen Annalen, Band 36, p. 49, verweist. Im letzten Abschnitt wird bei der Verallgemeinerung der "Regel über die elementare Schleife" die Zusammenordnung der Thetacharacteristiken für jedes bestimmte Schnittsystem gegeben.

Ich kann nicht umhin an dieser Stelle Herrn Prof. Dr. F. Klein für das hohe Interesse, das er dieser Arbeit genommen hat, so wie auch für die

wesentliche Förderung, die er durch seine guten Ratschläge der Abhandlung hat angedeihen lassen, meinen aufrichtigsten Dank zu sagen.

I.—VERSCHIEDENE SCHNITTSYSTEME.

§1.—*Die hyperelliptische Riemann'sche Fläche mit Schnittsystem.*

Die folgenden Seiten werden die hyperelliptische Riemann'sche Fläche behandeln. Es wird daher von Nutzen sein, wenn wir hier eine kurze Beschreibung derselben geben, indem wir dabei Riemanns "Grundlagen" §5 und seinen "Abel'schen Functionen" folgen. Die hyperelliptische Riemann'sche Fläche besteht aus einer zweiblättrigen Gauss-Ebene, deren beide Blätter eine gerade und endliche Anzahl von Punkten, sogenannte *Verzweigungspunkte*, gemeinsam haben. Diese Punkte sind durch Linien verbunden, in denen die beiden Blätter sich gegenseitig durchdringen, jedoch so, dass sie keinen anderen Punkt als die Endpunkte gemeinsam haben. Diese Linien werden *Uebergangslinien* genannt und so genommen, dass von jedem Verzweigungspunkte eine ungerade Zahl ausgeht; auch müssen diese Linien ohne Ausnahme in den Verzweigungspunkten endigen und entstehen, was augenscheinlich möglich ist, weil man eine gerade Anzahl Verzweigungspunkte hat. Die Figuren 1, 5, 10, u. s. w. stellen solche hyperelliptische Riemann'sche Flächen dar. Die Ziffern resp. die griechischen Buchstaben, bezeichnen die Verzweigungspunkte und die dieselben verbindenden Linien, die Uebergangslinien; die Linien in dem einen Blatt (dem sogenannten "Oberblatt") sind ausgezogen, die in dem anderen (dem "Unterblatt") sind punktirt. Wenn eine solche Fläche $(2p+2)$ Verzweigungspunkte hat, so sagt man, sie sei $(2p+1)$-fach zusammenhängend, d. h. auf der Fläche können von irgend einem Punkte zu einem anderen $(2p+1)$ in ihrem Wesen verschiedene Linien gezogen werden; und eine Linie ist *in ihrem Wesen verschieden* von einem Aggregat anderer, wenn sie durch ununterbrochenes Fortbewegen über die Fläche mit irgend einer dieser anderen oder einigen dieser anderen in ihrer ganzen Länge nicht zur Coincidenz gebracht werden und man auch nicht erreichen kann, dass sie aus einer Anzahl doppelter Drehungen um die einzelnen Verzweigungspunkte besteht, denn jede dieser doppelten Drehungen kann sich auf dem Verzweigungspunkte zusammenziehen. Es ergibt sich sofort aus dieser letzten Eigenschaft, dass jede geschlossene Linie auf der Fläche—wir wollen sie Schleife nennen—nicht in ihrem Wesen verschieden ist von der geschlossenen

Linie, welche überall gleich liegt, aber auf dem anderen Blatt; denn die eine kann ununterbrochen über die Fläche fortbewegt werden, bis sie mit der anderen coincidiert, mit Ausnahme von doppelten Drehungen um jeden eingeschlossenen Verzweigungspunkt. Um den Zusammenhang zwischen zwei solchen Schleifen zu erhalten, wollen wir nun die typischen Veränderungen, wie sie in den Figuren 1, 2, 3, 4, angedeutet sind, ins Auge fassen.

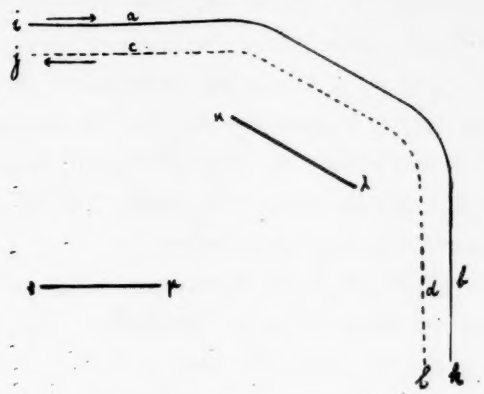

FIG. 1.

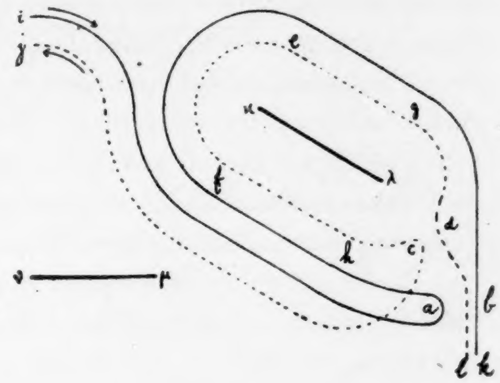

FIG. 2.

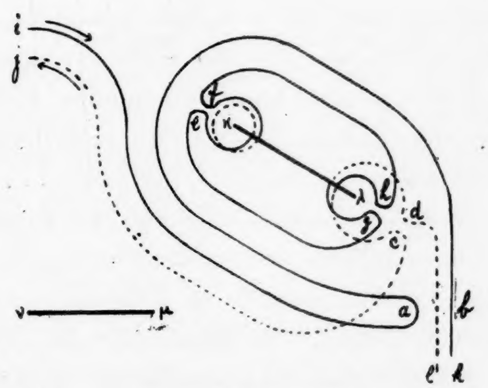

FIG. 3.

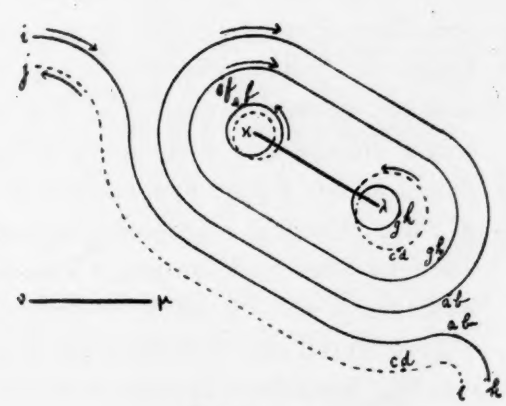

FIG. 4.

Gemäss der Definition der Fläche muss jede geschlossene Linie eine gerade Zahl von Verzweigungspunkten einschliessen. Wenn man erstens nur die Teile der beiden Linien ins Auge fasst, welche um ein Paar Punkte gehen, so wird sich zeigen, dass diese beiden Teile sich nur um zwei doppelte Drehungen um die einzelnen eingeschlossenen Verzweigungspunkte unterscheiden; ebenfalls kann

man zeigen, dass die übrigen Teile dieser beiden Linien in ihrem Wesen nicht verschieden sind, wenn man aufeinander folgend die Teile beobachtet, wie sie um die Verzweigungspunktpaare sich schlingen. Die in den Figuren 1, 2, 3, 4 angedeuteten Veränderungen zeigen, dass die Teile zwischen a und b, c und d in Figur 1 in ihrem Wesen dieselben sind. Aus Figur 1 erhält man Figur 2, wenn man die Teile in a und c beziehungsweise nach b und d bringt. Wenn man darauf f und h und e und g dieser Figur zwischen $\varkappa$ und λ auf ihre Plätze in Figur 3 bringt, so erhält man diese letztere Figur; und daraus entsteht Figur 4, wenn die Linien an den Punktpaaren a, b; c, d; e, f; g, h zusammengebracht und darauf wieder getrennt werden. Ausserdem ist es augenscheinlich, dass, wenn man eine dieser Linien aus der anderen durch fortwährende Fortbewegung über die Fläche entstehen lässt, die positive Richtung der Linie in dem einen Blatt die negative Richtung der anderen Linie auf dem anderen Blatt angibt.

Zum späteren Gebrauche wird diese $(2p+1)$-fach zusammenhängende Fläche einfach zusammenhängend gemacht werden. Dieses geschieht durch Riemanns *canonisches Schnittsystem*, oder einfach, *Schnittsystem*. Solch ein Schnittsystem wird durch jede beliebige p in ihrem Wesen verschiedene "primitive Schnittpaare" gebildet, sobald jedes Paar durch eine einfache Linie mit einem beliebig angenommenen gemeinsamen Punkte verbunden wird. Ein *primitiver Schnitt* ist jede geschlossene Linie auf der $(2p+1)$-fach zusammenhängenden Fläche, welche sich nie selbst und nur ein einziges Mal einen einzigen anderen Schnitt kreuzt, und welche daher in ihrem Wesen verschieden von einem Punkte ist. Zwei solche Schnitte, die sich kreuzen bilden ein *primitives Paar*. In den Figuren sind diese Paare bezeichnet mit $A_1, B_1; \ldots; A_k, B_k; \ldots; A_p, B_p$. Umgeht man einen Schnitt eines Paares, so kommt man von der einen Seite des zweiten Schnitts des Paares auf die andere Seite, und da die Functionen, welche auf der Fläche in Betracht kommen, Integrale sind, so entsteht eine *Periode* durch das Überschreiten des zweiten Schnittes. Es muss hier ein für alle Mal festgesetzt werden, was man unter positiver Richtung des *Umgehens* oder *Ueberschreitens* eines Schnittes verstehen soll. Die Periode *am* Schnitt A_k soll bezeichnet werden durch ω_k, an B_k durch ω_{k+p}. Diese Linien werden Schnitte genannt, weil durch sie die Fläche so geteilt werden kann, dass sie einfach zusammenhängend ist; aber wenn, was auf den folgenden Seiten gewöhnlich der Fall ist, und was wir besonders ins Auge fassen wollen, diese Linie *um* gewisse Verzweigungspunkte herumgeht, so nennen wir, wie oben, die Linie eine *Schleife*.

Ein Schnittsystem dieser oben beschriebenen Art ist das in Figur 5 gegebene, wo die Richtung, in welcher man die verschiedenen Schnitte zieht, die Benennung der positiven Seiten der Schnitte und das positive *Ueberschreiten* derselben mit den von Riemann gegebenen Regeln übereinstimmt* (Gesammelte Werke, SS. 122; 82, Fussnote; 98).

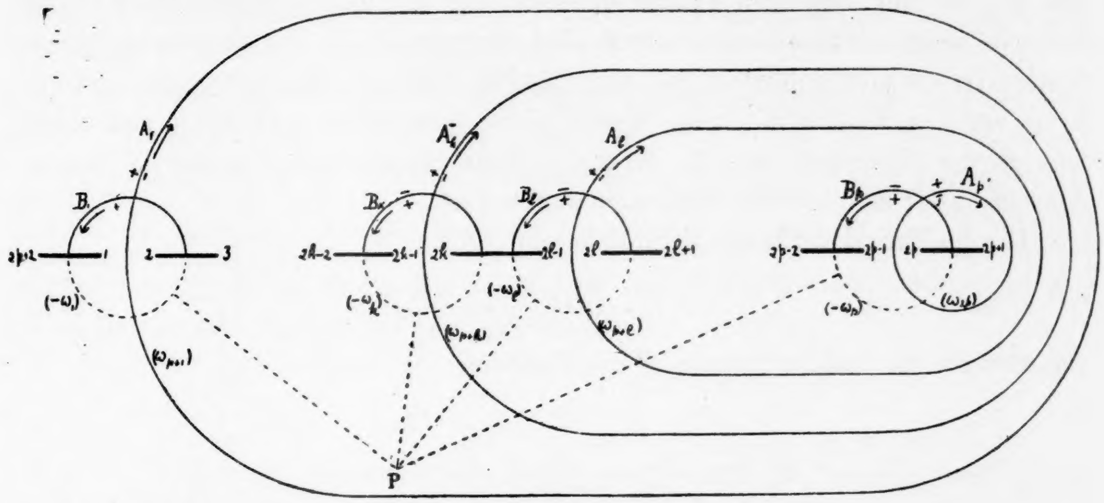

FIG. 5.

Da dieses Schnittsystem das geometrische Bild von den von Weierstrass† gebrauchten Integral-perioden ist, so erscheint es angemessen, bei der Gewohnheit zu bleiben und dasselbe als das Weierstrass'sche Schnittsystem zu bezeichnen, obwohl Weierstrass selbst niemals von Schnittsystemen spricht.

§2.—*Abel'sche Substitutionen.*

Wie man aus der Definition eines canonischen Schnittsystems ersehen kann, gibt es eine unendliche Anzahl solcher Schnittsysteme, wie wir sie hier nach

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* Diese Figur unterscheidet sich von der von Prof. Prym angewandten (Zur Theorie der Functionen in einer zweiblättrigen Fläche, Zürich, 1866) durch die Vertauschung von A_k B_k und die in Folge davon sich ergebende Vertauschung von positiver und negativer Richtung nach Riemann'schen Regeln. Prof. J. Thomae in Crelle's Journal, Bd. 71, S. 204 lässt die hier mit A bezeichneten Schnitte in einer der Riemann'schen entgegengesetzten Richtung entstehen. Dr. Burkhardt in den Mathematischen Annalen, Bde. 32 und 35, nimmt als positive Richtung des *Ueberschreitens* eines Schnittes die entgegengesetzte von der von Riemann gebrauchten an. Und Dr. J. Schröder vertauscht Riemanns Entstehungsrichtung der Schnitte in seiner Göttinger Dissertation 1892, S. 6.

† Cf. Über die Theorie der Abel'schen Functionen, Crelle's Journal, Bd. 47, S. 289 (1853), und Bd. 52, S. 285 (1856).

Riemann beschrieben, für jede Riemann'sche Fläche. Das Problem, wie, wenn ein Schnittsystem gegeben ist, alle anderen erhalten werden, ist die geometrische Formulirung des arithmetischen Problems der linearen ganzzahligen Transformation der Perioden. Um zu zeigen, dass diese beiden Probleme in der That dieselben sind, wollen wir in der üblichen Weise zwei beliebige Schnittsysteme S und S', die auf derselben Fläche übereinander liegen, ins Auge fassen. Die Schnitte, welche S zusammenersetzen, sind nach der Definition eines canonischen Schnittsystems alle in ihrem Wesen verschieden, während jede andere geschlossene Linie auf der Fläche in ihrem Wesen nicht verschieden sein kann von einer Summe der Elemente von S, weshalb jeder Schnitt des Systems S' einem Aggregate der Schnitte von S gleichbedeutend ist.

Die $2p$ Perioden für das Schnittsystem S seien:

$$P_1, \ldots, P_\lambda, \ldots, P_{2p},$$

während für das andere System S' die Perioden:

$$P'_1, \ldots, P'_\kappa, \ldots, P'_{2p}$$

sind. Dann entsteht die Periode an irgend einem Schnitt A'_κ (oder B'_κ) von S' dadurch, dass man den anderen Schnitt des Paares B'_κ (oder A'_κ) umgeht, welcher gleichbedeutend ist mit einer Combination von Schnitten von S, und jedes Umgehen eines Schnittes von S ergibt eine Periode an dem anderen Schnitt dieses Paares. Man kann dies ausdrücken durch die Formel:

$$P'_\kappa = \sum_1^{2p}{}_\lambda\, c_{\kappa\lambda} P_\lambda, \quad (\kappa = 1, 2, \ldots, 2p),$$

wo jedes $c_{\kappa\lambda}$ eine ganze Zahl ist. Diese Substitution kann auch in die Form eines Schemas ausgedrückt werden, dass man die Coefficienten $c_{\kappa\lambda}$ in die Form einer Determinante mit $2p$ Zeilen und $2p$ Colonnen bringt wo die λten, bez. $(\lambda + p)$ten Coefficienten in der κten Zeile die Coefficienten der Perioden am λten Schnittpaare von S sind.

Aber es sind gewisse Beziehungen zwischen den Constanten $c_{\kappa\lambda}$, so dass jedem Wechsel eines Schnittsystems nicht eine allgemeine lineare ganzzahlige Substitution entspricht, sondern eine solche, welche die beiden folgenden Bedingungen erfüllt:

1). Die Transformationsdeterminante ist gleich der Einheit; denn in der

vorhergegangenen Erörterung der Beziehungen zwischen S und S' können die Symbole P' und P vertauscht werden, und wenn dann die Gleichungen:

$$P'_\kappa = \sum_1^{2p}{}^\lambda c_{\kappa\lambda} P_\lambda,$$

gelöst sind, so müssen die Perioden P als lineare ganzzahligen Functionen der Perioden P' ausgedrückt sein.

2). Da die sogenannten Riemann'schen bilinearen Relationen zwischen den Perioden verschiedener Integrale nicht von der Form des Schnittsystems abhängen, so sind diese Beziehungen invariant, und daher ist die Determinante gleich der positiven Einheit und die Constanten $c_{\kappa\lambda}$ müssen den $\frac{2p(2p-1)}{2}$ Relationen genügen, denen Kronecker in seinen Vorlesungen, 1864–65, den Namen Abel'sche Relationen gab. Es sind:

$$\sum_1^p{}^\rho (c_{\rho\alpha} \cdot c_{\rho+p,\,\beta} - c_{\rho\beta} \cdot c_{\rho+p,\,\alpha}) = \begin{cases} 1, & \text{wenn } \alpha + p = \beta \\ 0, & \text{wenn } \alpha + p \gtrless \beta \end{cases}$$

mit $\alpha < \beta$; $\beta = 2, 3, \ldots, 2p$; $\alpha = 1, 2, \ldots, \beta - 1$.

Diese Untergruppe linearer ganzzahliger Substitutionen mit der Determinante $= +1$, welche den $\frac{2p(2p-1)}{2}$ Abel'schen Relationen genügt, wird von *M.* C. Jordan die Abel'sche Gruppe genannt und die Substitutionen der Gruppe die *Abel'schen Substitutionen.** Es entspricht nicht nur jedem Wechsel des Schnittsystems eine Abel'sche Substitution, sondern es ist auch wiederum wahr, dass jede Abel'sche Substitution einen Wechsel des Schnittsystems erzeugt. Die Anzahl der "erzeugenden"† Substitutionen der Abel'schen Gruppe (d. h. derjenigen, durch deren wiederholte Anwendung alle anderen abgeleitet werden können) kann stets auf fünf zurück geführt werden, wie Prof. Krazer‡ bewiesen hat; und jüngst hat Dr. Burkhardt|| gezeigt, dass diese Anzahl für $p = 2$, auf zwei, und für $p > 2$, wenigstens auf drei gebracht werden kann. Wenn man

* Camille Jordan, Traité des substitutions, Paris, 1870, p. 172.

† Kronecker, Berliner Monatsberichte, 1866; cf. Clebsch und Gordan, Abel'sche Functionen, 1866, S. 308; Henoch, Berliner Dissertation, 1867.

‡ A. Krazer, Annali di Matematica pura ed applicata, Serie IIª, Tomo XII°, 1884.

|| H. Burkhardt, Göttinger Nachrichten, 1890, S. 381.

die durch die erzeugenden Substitutionen bedingten Operationen geometrisch ausführt, so ist es klar, dass eine jede zu einem canonischen Schnittsystem führt und daher jede Abel'sche Substitution ein canonisches Schnittsystem gibt. Dies ist die von Prof. J. Thomae in Crelle's Journal, Band 75, eingeführte Methode, wo er zeigt, dass das Gleiche auch für eine weit grössere Zahl von erzeugenden Substitutionen zutrifft. Cf. auch Burkhardt, Systematik der hyperelliptischen Functionen, §7, Mathematische Annalen, Band 35, S. 212.

§3.—*Monodromiesubstitutionen.*

Die Methode, nach welcher die Riemann'sche Fläche erzeugt wurde, behauptete nichts in Betreff der Unbeweglichkeit der Verzweigungspunkte. Wenn man nun eine hyperelliptische Riemann'sche Fläche mit einem canonischen Schnittsystem nimmt, und die Verzweigungspunkte nach Belieben über die Fläche fortbewegt, so müssen auch die Schleifen als sich bewegend angenommen werden; denn die Schleife, welche in ihrem Wesen verschieden von gewissen anderen Linien ist, wird dadurch bestimmt, dass sie um einen Verzweigungspunkt geht, so dass, wenn ein Verzweigungspunkt in seiner Bewegung sich einer Schleife nähert, diese Schleife vor dem Verzweigungspunkte über die Fläche zurückweichen muss. Da die Uebergangslinien dadurch bestimmt sind, dass sie in Verzweigungspunkten entstehen, so muss der Verzweigungspunkt bei der Bewegung die in ihm entstehenden Uebergangslinien nach sich ziehen, kann aber irgend eine Uebergangslinie überschreiten. Diese Bewegung der Verzweigungspunkte, Uebergangslinien, und Schnitte wird so lange fortgesetzt, bis jeder Verzweigungspunkt eine Lage einnimmt, die ebenfalls vorher von einem Verzweigungspunkt eingenommen wurde, so dass die Lage der Verzweigungspunkte schliesslich dieselbe ist, obwohl eine Permutation zwischen einigen von ihnen stattgefunden haben mag. Es ist ersichtlich, dass während der Bewegung keine zwei Verzweigungspunkte zusammenfallen dürfen. Wenn die Verzweigungspunkte ihre endgültige Lage eingenommen haben, so kann das System der Uebergangslinien verschieden sein von dem ursprünglichen; aber es kann, wenn man will, immer vereinfacht, und in die ursprüngliche Form gebracht werden, wenn man die sogenannten "oberen" und "unteren" Blätter an gewissen in sich zurücklaufenden Linien vertauscht. Z. B. sind zwischen den Figuren 6 und 7 die Blätter an einer Linie vertauscht worden, welche von x ausgeht und in x endigt und um λ, μ, ν (und ξ in II) herumgeht; wäre der Wechsel nicht gemacht, so

wäre der einzige Unterschied der gewesen, dass in der Figur 7 und den folgenden, drei Uebergangslinien anstatt einer in dem Punkte x entstehen würden. So ist die hyperelliptische Irrationalität dieselbe wie die ursprüngliche, aber das Schnittsystem hat sich geändert. Von zwei solchen Schnittsystemen sagt man, dass sie *gleichartig* sind; dass das letztere Schnittsystem aus dem ersteren durch *Monodromie* der Verzweigungspunkte entstanden ist. Und die Abel'sche Substitution, welche diese Änderung ergibt, ist *M.* Camille Jordans *Monodromie-substitution.* Diese Monodromie der Verzweigungspunkte gibt *M.* Jordan in seinen "Substitutions," SS. 339 und folgende, aber schon vorher war diese Bewegung der Wurzeln einer Gleichung und der Rückkehr zu demselben Werthe von Hermite in den "Comptes rendus," t. 32 (1851), S. 458, einer Betrachtung unterzogen worden.

§4.—*Monodromie ohne Permutation.*

Betrachten wir zuerst die Monodromie ohne Permutation, die also, bei der jeder einzelne Verzweigungspunkt in seine eigene Lage zurückkehrt. In diesem Falle besteht jeder Schnitt des zweiten Schnittsystems aus der ursprünglichen Form dieses Schnittes in dem ersten Schnittsystem, sowie aus einer zweifachen linearen Combination von Schnitten des ersten Systems.

Die Wahrheit dieser Behauptung ergibt sich, wenn man einen Verzweigungspunkt über die Fläche wanderen lässt, und man, sobald er in seine ursprüngliche Lage zurückgekehrt ist, die Form des Teiles der Schleife beobachtet (in den Figuren 6, 7, 8, 9 den Teil von e nach f) den der Verzweigungspunkt vor sich hin über die Fläche geschoben hat. Man wird finden, dass solch eine Schleife in jedem Falle der ursprüngliche Teil der unveränderten Schleife ist, oder dieser Teil zusammengenommen mit zwei in sich zurücklaufenden Linien auf der Fläche, welche überall neben einander liegen, aber auf verschiedenen Blättern, welche auf den beiden Blättern entgegengesetzte Richtung haben, und deshalb gleich dem Zweifachen einer dieser Linien sind (§1). Diese letztere Linie besteht, weil sie eine in sich zurücklaufende Linie auf der Fläche ist, aus einer linearen Combination von Schnitten des ersten Schnittsystems, und der Schnitt des zweiten Systems wird daher gebildet von dem correspondirenden Schnitt des ersten Systems und einer doppelten linearen Combination von Schnitten dieses Systems.

Jeder Verzweigungspunkt muss bei der Hin- und Rückbewegung entweder um eine gerade oder ungerade Anzahl anderer Verzweigungspunkte gehen, und

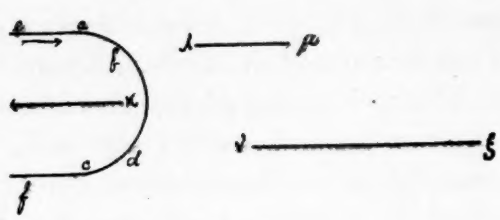

FIG. 6I.

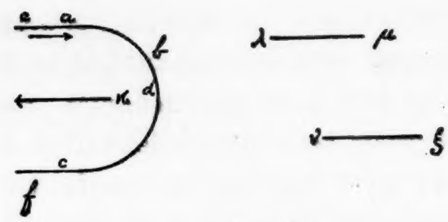

FIG. 6II.

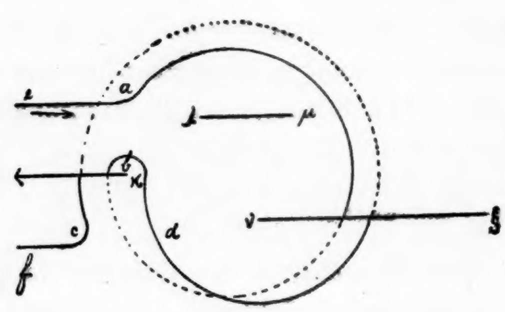

FIG. 7I.

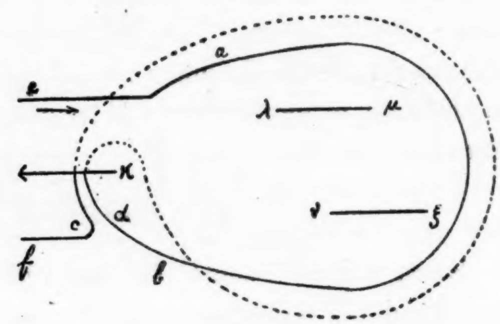

FIG. 7II.

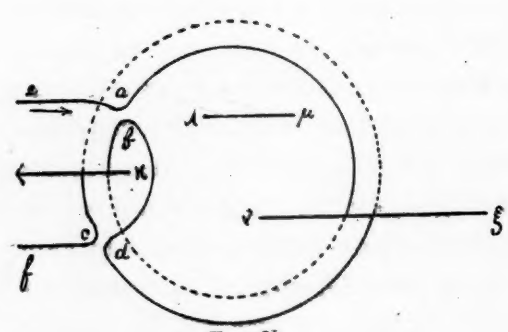

FIG. 8I.

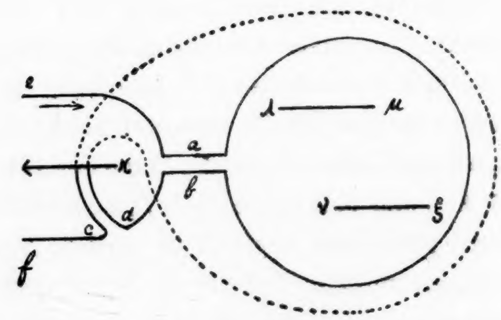

FIG. 8II.

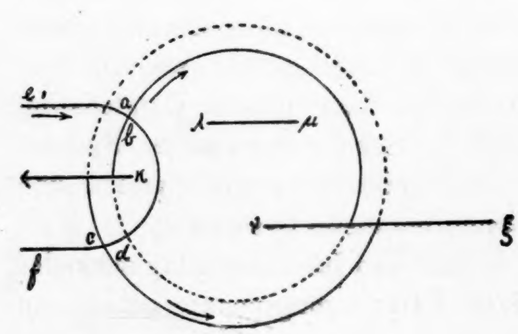

FIG. 9I.

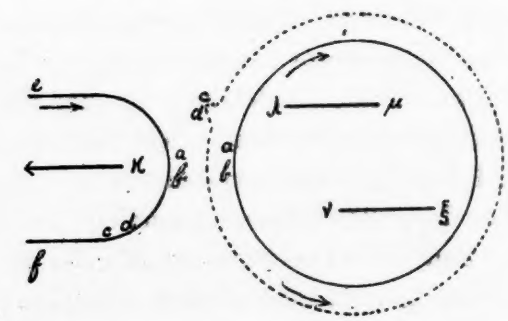

FIG. 9II.

die möglichen correspondirenden Änderungen sind in den Figuren 6, 7, 8, 9 dargestellt.

Figur 7 entsteht aus Figur 6 wenn der Verzweigungspunkt $\varkappa$ nachdem er um λ, μ, ν oder λ, μ, ν, ξ herumgegangen ist in seine ursprüngliche Lage zurückkehrt; in Figur 8 sind die Schleifen der Figur 7 zusammengequetscht, da verschiedene Teile einander in a und b und in c und d genähert sind; und in Figur 9 sind a und b, c und d zur Coincidenz gebracht (und in 9^{II} wieder getrennt). Da dieser Schnitt in dem zweiten System der ursprüngliche Schnitt des ersten Systems zusammen mit einer doppelten linearen Combination von diesen Schnitten ist, so kann die Periode $P'_\varkappa$ an dem anderen Schnitt des Paares so ausgedrückt werden:

$$P'_\varkappa = P_\varkappa + 2\sum_{1}^{2p}{}^{\lambda} c'_{\varkappa\lambda} P_\lambda = (1 + 2c'_{\varkappa\varkappa})P_\varkappa + 2\sum_{1}^{\varkappa-1}{}^{\lambda} c'_{\varkappa\lambda} P_\lambda + 2\sum_{\varkappa+1}^{2p}{}^{\lambda} c'_{\varkappa\lambda} P_\lambda,$$

oder wenn $c_{\varkappa\lambda}$ der ganze Coefficient in der Substitutionsdeterminante ist:

$$\begin{vmatrix} c_{11}, & c_{12}, & \ldots, & c_{1,2p} \\ c_{21}, & c_{22}, & \ldots, & c_{2,2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2p,1}, & c_{2p,2}, & \ldots, & c_{2p,2p} \end{vmatrix}, \quad c_{\varkappa\lambda} \equiv \begin{cases} 1 \pmod 2, \text{ wenn } \varkappa = \lambda, \\ 0 \pmod 2, \text{ wenn } \varkappa \gtrless \lambda. \end{cases}$$

Solch eine Substitution wo die Coefficienten gerade sind, ausser in der Hauptdiagonale, wo sie ungerade sind, heisst congruent (modulo 2) zur Identität. Und es ist so gezeigt worden, dass Monodromie ohne Permutation eine Substitution erzeugt, welche congruent (modulo 2) zur Identität ist. Es ist oft vorteilhaft statt der oben betrachteten Abel'schen Substitutionen Congruenzen (modulo 2) anzuwenden; und die Ordnung der Gruppe dieser Abel'schen Congruenzen (modulo 2) ist:

$$\Omega_2 = (2^{2p} - 1)2^{2p-1}(2^{2p-2} - 1) \ldots . \; 2^3(2^2 - 1)2.$$

In der That ist dies die allgemeine Formel, welche *M.* Camille Jordan* in Beantwortung der Frage gegeben hat: Wie viele Systeme incongruenter (modulo 2) Zahlen $c_{\varkappa\lambda}$ gibt es, welche den von einer Abel'schen Substitution geforderten Bedingungen genügen? Oder mit anderen Worten für diesen besonderen Fall: Wie viele verschiedene Transformationsdeterminanten gibt es von der oben

* "Substitutions," p. 174.

erwähnten Form, wo die Abel'sche Relationen zwischen den Coefficienten zutreffen, und wo ausserdem die Coefficienten nur gleich Null und der Einheit sind? *

Wiederum aber erhält man jede Substitution, welche zur Identität congruent (modulo 2) ist durch Monodromie ohne Permutation, wie *M.* Jordan gezeigt hat, indem er bewies, dass eine derartige Substitution aus gewissen erzeugenden Substitutionen hervorgeht, bei denen die Voraussetzung zutrifft.

Diese erzeugenden Substitutionen sind zweierlei Art:† Erstens die, bei der der eine Schnitt zweimal dem anderen Schnitt des Paares hinzugefügt ist, und zweitens die, wo zu einem Schnitt eines Paares zweimal ein Schnitt eines anderen Paares, und zugleich zu dem anderen Schnitt dieses zweiten Paares minus zweimal der zweite Schnitt des ersten Paares hinzugefügt sind. Diese beiden Arten erzeugender Substitutionen sind einerseits von dem Typus, dessen geometrisches Gegenstück in den Figuren 9^{I}, 8^{I}, 7^{I}, 6^{I} gegeben ist; und andererseits von dem Typus der in der Reihe der Figuren von 9^{II} bis zu 6^{II} hinab dargestellt ist; aber im letzteren Falle ist nur die Änderung in einem der beiden Schnitte gegeben worden.

§5.—*Anzahl der verschiedenartigen Schnittsysteme.*

Es ist oben gesagt worden, dass zwei Schnittsysteme gleichartig sind, wenn das eine aus dem anderen durch Monodromie hervorgeht. Von zwei Schnittsystemen die nicht in obiger Beziehung stehen kann man sagen, dass sie *verschiedenartig* sind. Solche gleichartige und verschiedenartige Schnittsysteme wurden zuerst von Prof. J. Thomae untersucht. Die Zahl der verschiedenartigen

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* Herr Dr. H. Burkhardt hat die Güte gehabt, mich darauf aufmerksam zu machen, dass Galois in seinem Briefe an *M.* August Chevalier vom 29. Mai 1832, welcher zuerst in der Revue encyclopédique, September 1832, später in Liouville's Journal, tome XI, 1846, auf Seite 414 veröffentlicht ist, als Ordnung der Gruppe nicht diesen Werth von Ω sondern den folgenden gibt, welcher für jede Primzahl p die Ordnung der allgemeinen linearen Gruppe ist:

$$\Omega=(p^{2n}-1)(p^{2n}-p)\ldots.(p^{2n}-p^{2n-1}).$$

Es ist dies insofern interessant, als es zeigt, dass Galois keinen Begriff von der Rolle hatte, welche die bilinearen Relationen in der Theorie der Transformation spielen. Wenn jedoch die Thetafunctionen mit dem Verhalten der τ's ins Auge gefasst werden, so werden diese Relationen notwendig gebraucht; und so sind auch diese Relationen für den hyperelliptischen Fall in Weierstrass's Braunsberger Programm (1849) und, für $p=2$, in Rosenhains Artikel über die Thetafunctionen (1851) gegeben.

† Cf. Jordan, "Substitutions," p. 360; Burkhardt, Systematik der hyperelliptischen Functionen, Mathematische Annalen, Band 35, p. 235.

Schnittsysteme ist von Herrn Prof. F. Klein in der folgenden Weise bestimmt worden: Monodromie, in welcher jeder Verzweigungspunkt in seine ursprüngliche Lage zurückkehrt, gibt eine Substitution, welche zur Identität congruent (modulo 2) ist, und deshalb ist die Zahl der Schnittsysteme, welche aus dieser Monodromie nicht entstehen können, gleich Ω_2, die oben angegebene Zahl. Wenn man auch die Monodromie in Betracht zieht, bei welcher jeder Verzweigungspunkt endlich in eine früher von einem anderen Verzweigungspunkte angenommene Lage kommt, so sind, da die Zahl solcher Permutationen der $(2p+2)$ Verzweigungspunkte gleich $(2p+2)!$ ist, und da jede Monodromiesubstitution durch die zwei Operationen erlangt wird, dass erstens Monodromie der Verzweigungspunkte mit Permutation $(2p+2)!$ Ansätze gibt, und dass zweitens Monodromie ohne Permutation eine Substitution ergibt, die zur Identität congruent (modulo 2) ist, die Monodromiesubstitutionen congruent (modulo 2) zu $(2p+2)!$ Repräsentanten und bestehen aus sämmtlichen Abel'schen Substitutionen, welche congruent (modulo 2) zu diesen Repräsentanten sind. D. h. die Anzahl der verschiedenartigen Schnittsysteme ist:

$$\frac{\Omega_2}{(2p+2)!} = \frac{(2^{2p}-1)2^{2p-1}(2^{2p-2}-1)\ldots\ldots 2^3(2^2-1)2}{(2p+2)!}.$$

Diese Zahl, welche mit zunehmendem p schnell anwächst, hat für die niederen Werte von p die Werte:

p	2	3	4	5
$\frac{\Omega_2}{(2p+2)!}$	1	36	13056	51806208

II.—Construction der verschiedenen Schnittsysteme.

§6.—*Schnittsysteme für jedes p.*

In Figur 5 ist ein Schnittsystem gegeben, welches für jedes p ein Typus sein wird. Ausserdem zeigt diese Figur den einzigen Typus eines Schnittsystems für den elliptischen Fall, wenn nur die Verzweigungspunkte $2p+2$, 1, 2, $2p+1$ und die Schnittpaare A_1, B_1 ins Auge gefasst werden; und für $p=2$, wenn die Verzweigungspunkte 3, 4 und das Schnittpaar A_2, B_2 hinzugenommen werden.

Weiter unten sind Beispiele für die sechs und dreissig Typen gegeben, welche

für $p=3$ vorhanden sind, und für diesen Fall wollen wir durch geometrische Betrachtung der Riemann'schen Fläche zeigen, dass es gerade sechs und dreissig Arten Schnittsysteme gibt. Ausser dem Schnittsystem in Figur 5 (wenn zwölf Verzweigungspunkte und fünf Schnittpaare angenommen werden) sind für $p=5$ noch zwei andere Schnittsysteme in §9 dargestellt, und in §10 sind die Transformationsdeterminanten gegeben, vermittelst deren man noch zweimal acht andere Schnittsysteme erhalten kann.

§7.—*Geometrische Characteristik.*

Es ist erwähnt worden, das ein canonisches Schnittsystem der zweiblättrigen Riemann'schen Fläche für $p=3$ aus drei Paar Schleifen besteht, bei denen jede Schleife eine gerade Anzahl von Verzweigungspunkten von der ganzen Zahl derselben trennt; und da in dem vorliegenden Falle nur acht von diesen Punkten vorhanden sind, so muss jede Schleife zwei oder vier Punkte enthalten, und soll dann zweipunktige oder vierpunktige Schleife genannt werden. Sei (1) das Zeichen für die erstere, (0) das für die letztere, dann wird ein Paar durch $\binom{0}{0}$, $\binom{0}{1}$, $\binom{1}{0}$ oder $\binom{1}{1}$ dargestellt, wo das obere Zeichen sich auf die Schleife B_i bezieht, das untere auf die Schleife A_i (cf. Figuren 10–15). Das Schnittsystem ist eine Combination von Paaren und kann bezeichnet werden durch:

$$\begin{pmatrix} g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{pmatrix}, \quad g_i,\ h_i = 1,\ 0;$$

welches wir die "Geometrische Characteristik" des Systems nennen, und welches der Einfachheit wegen durch $\binom{g}{h}$ bezeichnet sei. Hier bedeutet, $g_i=1$, dass B_i eine zweipunktige, $g_i=0$, dass es eine vierpunktige Schleife ist, und dasselbe trifft zu für h_i mit Bezug auf A_i. $\binom{g}{h}$ wird ungerade oder gerade genannt jenachdem:

$$g_1h_1+g_2h_2+g_3h_3 \equiv 1 \text{ oder } 0 \text{ (modulo 2)}.$$

Das $\binom{g}{h}$ des in Figur 5 gegebenen Weierstrass'schen Systems (wo acht Verzweigungspunkte und drei Schnittpaare genommen sind) ist durch $\binom{111}{101}$ dargestellt; wenn aber die einzelnen Schnitte anders benannt werden, so wird sich dieses $\binom{g}{h}$ ändern, wird aber aus einem Aggregate von drei Paaren bestehen,

von denen jedes Paar den Typus eines der ursprünglichen Paare hat. Wenn, z. B.:

$$\begin{aligned} A_1' &= B_2, & A_2' &= A_3, & A_3' &= B_1, \\ B_1' &= -A_2, & B_2' &= B_3, & B_3' &= -A_1, \end{aligned}$$

ist, so ist $\begin{pmatrix} g \\ h \end{pmatrix}$ für dieses abgeleitete Schnittsystem gleich $\begin{pmatrix} 011 \\ 111 \end{pmatrix}$.

Es soll nun gezeigt werden, dass ein ungerades $\begin{pmatrix} g \\ h \end{pmatrix}$ unmöglich ist. Betrachten wir ein erstes Paar des Typus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, dieses Paar schliesst drei Verzweigungspunkte ein und es gibt keine Schleife, welche einen dieser drei Punkte einschliesst, ohne sie alle einzuschliessen. Man nehme darauf eine Schleife des zweiten Paares des Typus (0). Dann kann keine vierpunktige Schleife gezogen werden, die nicht diese vierpunktige Schleife des zweiten Paares schnitte, d. h. das dritte Paar muss von dem Typus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ sein. Und wenn, vorausgesetzt dass das erste Paar dasselbe bleibt $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, ein zweites Paar $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ genommen wird, so wird dieses zweite Paar andere drei Punkte einschliessen und nur zwei Punkte uneingeschlossen lassen; es kann deshalb ein drittes Paar zweipunktiger Schleifen nicht gezogen werden. D. h., wenn ein Paar des Typus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ vorhanden ist, so muss ein zweites von demselben Typus sein, alle drei Paare können aber nicht von diesem selben Typus sein; mit anderen Worten, ein canonisches Schnittsystem für $p = 3$ muss ein gerades $\begin{pmatrix} g \\ h \end{pmatrix}$ haben. Es gibt sechs und dreissig gerade $\begin{pmatrix} g \\ h \end{pmatrix}$ und die Figuren 10 bis 15 zeigen, dass die correspondierenden Systeme alle vorhanden sind. Diese Zahl sechs und dreissig ist gleich der Zahl der verschiedenartigen Schnittsysteme, welche für $p = 3$ am Ende von §5 gefunden ist. Diese sechs und dreissig Characteristiken $\begin{pmatrix} g \\ h \end{pmatrix}$ können in die folgenden sechs Familien eingeteilt werden:

1) $\begin{pmatrix} 000 \\ 000 \end{pmatrix}$,

2) $\begin{pmatrix} 001 \\ 000 \end{pmatrix} \begin{pmatrix} 010 \\ 000 \end{pmatrix} \begin{pmatrix} 100 \\ 000 \end{pmatrix} \begin{pmatrix} 000 \\ 100 \end{pmatrix} \begin{pmatrix} 000 \\ 010 \end{pmatrix} \begin{pmatrix} 000 \\ 001 \end{pmatrix}$,

$$3)\ \binom{101}{000}\binom{110}{000}\binom{011}{000}\binom{000}{101}\binom{000}{110}\binom{000}{011},$$

$$\binom{001}{010}\binom{010}{100}\binom{100}{001}\binom{001}{100}\binom{010}{001}\binom{100}{010},$$

$$4)\ \binom{111}{000}\binom{000}{111}\binom{001}{110}\binom{010}{101}\binom{100}{011}\binom{011}{100}\binom{110}{001}\binom{101}{010},$$

$$5)\ \binom{101}{101}\binom{011}{011}\binom{110}{110},$$

$$6)\ \binom{111}{101}\binom{111}{011}\binom{111}{110}\binom{110}{111}\binom{101}{111}\binom{011}{111}.$$

§8.—*Die sechs und dreissig Systeme für $p = 3$.*

Die unendliche Zahl von Schnittsystemen kann gemäss dem $\binom{g}{h}$, zu dem sie gehören, in sechs und dreissig Classen eingeteilt werden und auch diese gruppieren sich in sechs Familien. Es erübrigt, einen Repräsentanten für eine jede dieser Familien zu geben, was in den Figuren 10 bis 15 geschehen ist, in denen ein Verzweigungspunkt im Unendlichen liegt, während die anderen dreifach durch die Uebergangslinien verbunden sind. Diese Figuren sind die stereographische Projection der Ecken und Kanten eines Würfels. Jede Schleife hat die

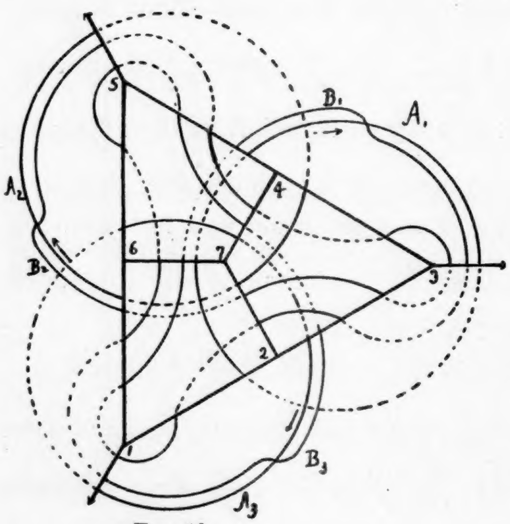

FIG. 10.

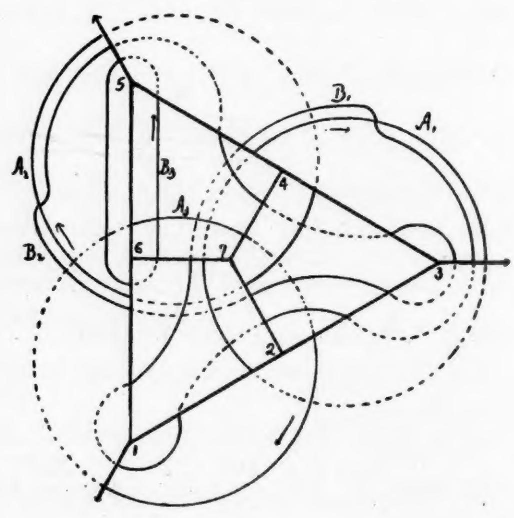

FIG. 11.

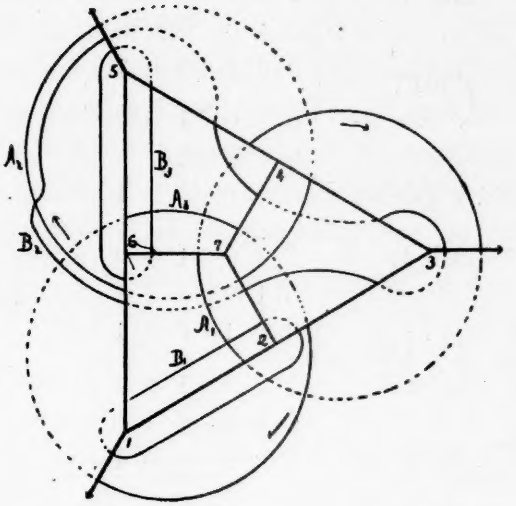

FIG. 12.

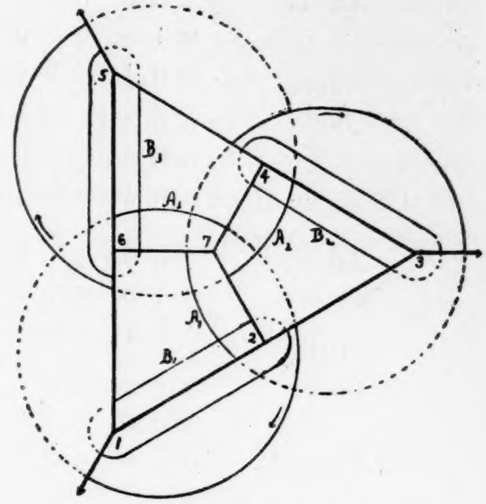

FIG. 13.

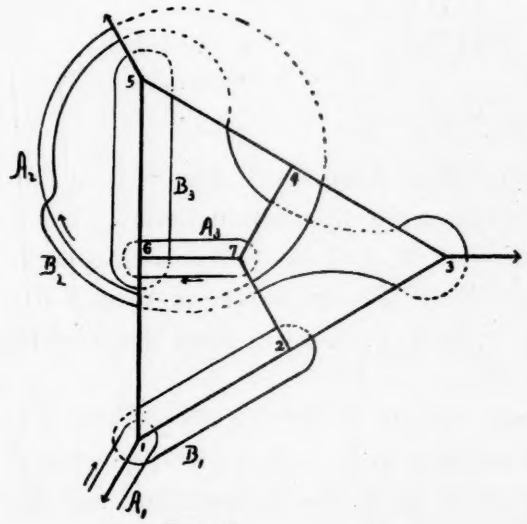

FIG. 14.

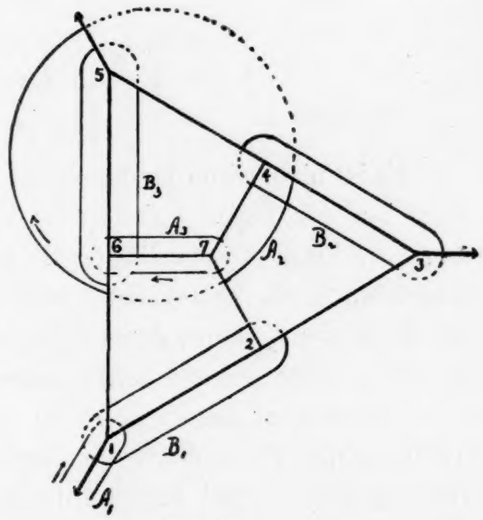

FIG. 15.

positive Richtung im Sinne eines Uhrzeigers, wenn die entgegengesetzte Richtung nicht durch einen Pfeil angezeigt ist. Die sechs gegebenen Schnittsysteme ergeben sich aus dem ersten durch die Anwendung der sechs beigefügten linearen Transformationen der Perioden, wo nach der Transformation die Endform der Schnitte zu der "einfachsten" im Sinne von §13 gemacht worden ist.

Ad 1) $\binom{000}{000}$ Cf. Figur 10

$$\begin{vmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{vmatrix}$$

Ad 2) $\binom{001}{000}$ Cf. Figur 11

$$\begin{vmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{vmatrix}'$$

Ad 3) $\binom{101}{000}$ Cf. Figur 12

$$\begin{vmatrix} 1&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{vmatrix}'$$

Ad 4) $\binom{111}{000}$ Cf. Figur 13

$$\begin{vmatrix} 1&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{vmatrix}'$$

Ad 5) $\binom{101}{101}$ Cf. Figur 14

$$\begin{vmatrix} 1&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&1\\ 0&0&1&1&0&1\\ 0&0&0&0&1&0\\ 1&0&0&1&0&1 \end{vmatrix}'$$

Ad 6) $\binom{111}{101}$ Cf. Figur 15

$$\begin{vmatrix} 1&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&0&1\\ 0&0&1&1&0&1\\ 0&0&0&0&1&0\\ 1&0&0&1&0&1 \end{vmatrix}'$$

Es ist nicht ohne Interesse, zu bemerken, dass diese letzte Figur 15 in der That mit der Figur 5 (mit acht Verzweigungspunkten) übereinstimmt. Denn wenn in Figur 15 die Blätter längs der durch 8, 3, 2, 7, 4, 5, 8 in sich zurücklaufenden Linie vertauscht werden, so bleibt die Fläche unverändert und das ist die Figur 5, wenn die Linie durch 8, 1, 2, 3, 4, 5, 6, 7, längs der reellen Achse zu einer geraden Linie gestreckt wird.

Das System der Figur 13 ist von Herrn Prof. H. A. Schwarz auf Tafel III, 10 in seiner Preisschrift der Berliner Akademie vom 4. Juli 1867, gegeben;* da aber der Würfel vom Centrum einer Seite als Projectionscentrum auf die Ebene projicirt worden ist, so ist die dort gegebene Figur nicht symmetrisch.

§9.—*Symmetrische Schnittsysteme für* $p = 5$.

Wie wir oben gesehen haben, wächst die Zahl der verschiedenartigen Schnittsysteme rasch an sobald p grösser als drei ist, und diese Zahl wird immer grösser sein als die Zahl der transcendentalen Characteristiken, welche mit der

* Cf. H. A. Schwarz, Gesammelte Abhandlungen, Figur 52, S. 116.

algebraischen Characteristik $\phi_0 . \psi_{2p+2}$ oder $\phi_1 . \psi_{2p+1}$ für verschiedene Schnittsysteme coordinirt werden können. Und sobald p grösser als drei ist, werden die die Schnittsysteme darstellenden Figuren so compliciert, dass das Gewirr der Linien eher Unklarheit bringt als Klarheit; man sieht dies sofort, wenn man die am wenigsten complicierte der durch die Determinanten des nächsten Paragraphen angedeuteten Operationen ausführt. Deshalb werden hier für $p = 5$ nur zwei einfache Systeme gegeben, die zu derselben transcendentalen Characteristik $\begin{pmatrix}11111\\11111\end{pmatrix}$ gehören (*vide* Figuren 16 und 17).

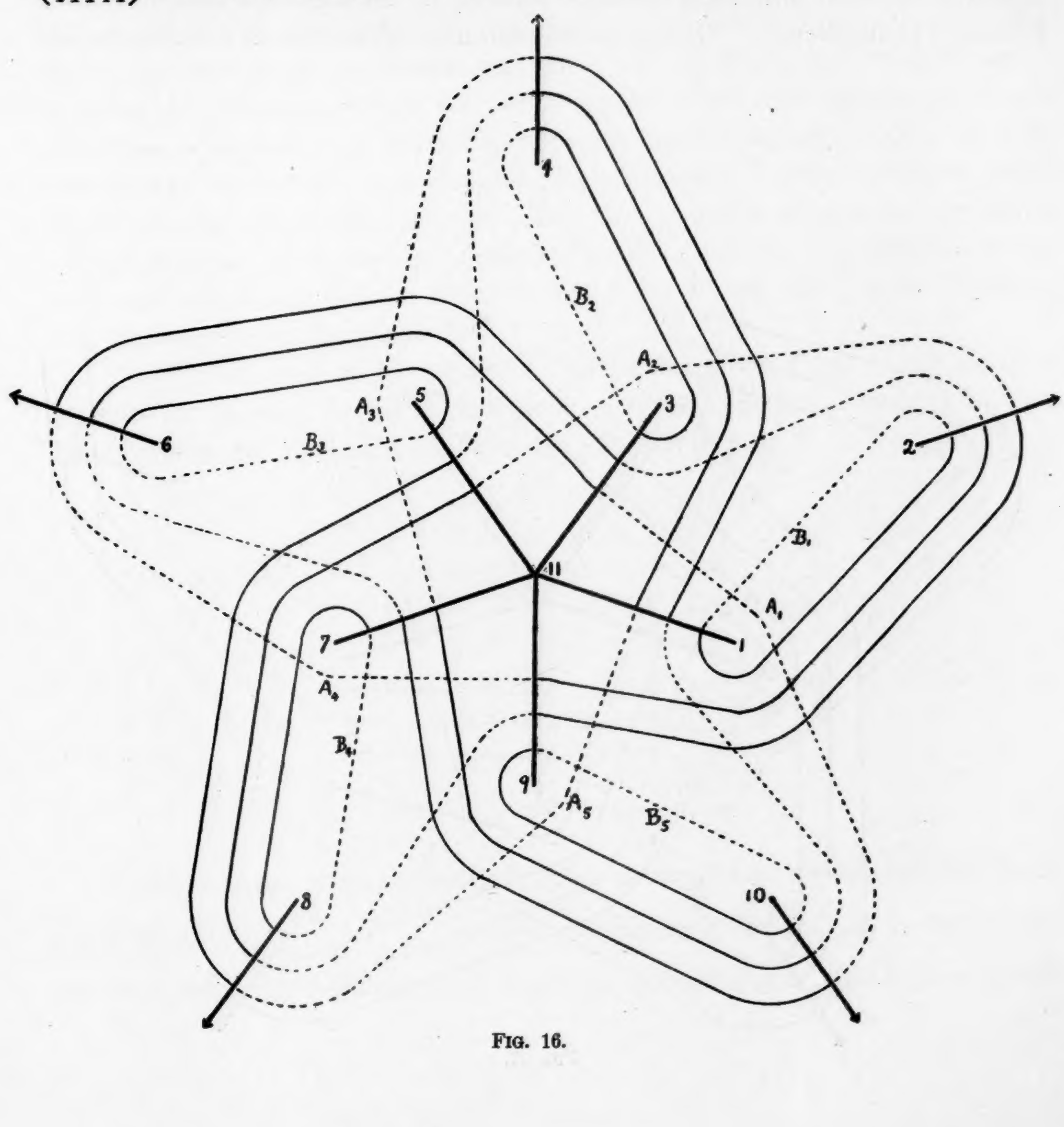

FIG. 16.

Jedes Element der im §7 definirten geometrischen Characteristik ist zur Anzahl der in der zugeordneten Schleife des Schnittsystems eingeschlossenen Paare von Verzweigungspunkten congruent (modulo 2). Weil bei den zum Falle $p=3$ zugehörigen acht Verzweigungspunkten nur zweipunktige und vierpunktige Schleifen vorkommen, daher zeigt in diesem Falle unzweideutig das Element der Characteristik die Anzahl der in der correspondirenden Schleife eingeschlossenen Verzweigungspunkte. Es giebt aber für $p=5$ zum Beispiel zweipunktige, vierpunktige und sechspunktige Schleifen, und so wird nicht nur eine zweipunktige sondern auch eine sechspunktige Schleife in der Characteristik durch ein Element (1) angedeutet. Da also zu mehreren Schnittsystemen dieselbe geome-

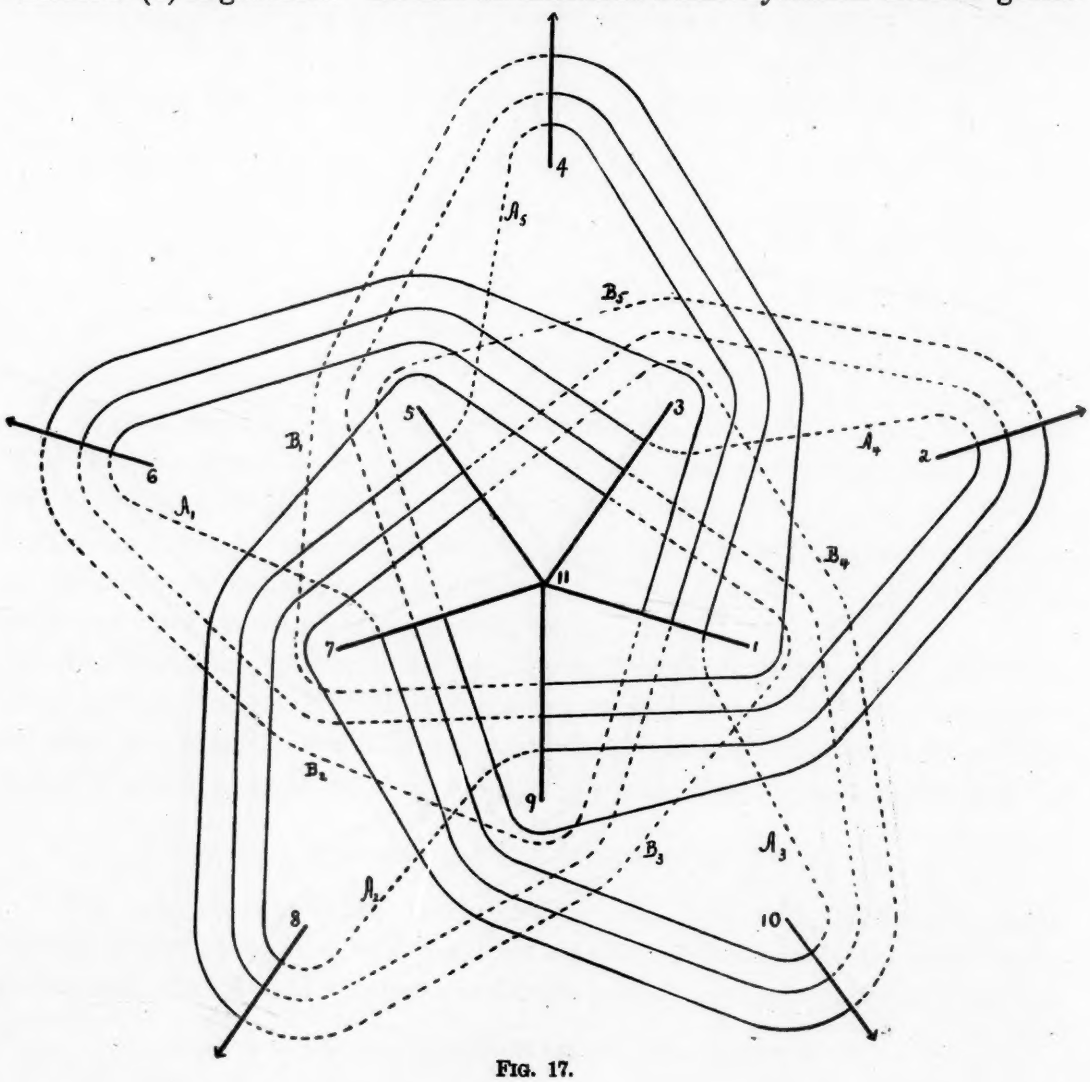

FIG. 17.

trische Characteristik coordinirt ist, so wird es nicht vorteilhaft sein, eine Ausdehnung jener Überlegung auf die höheren Fälle durchzuführen.

Da es für $p = 5$ nur 496 Characteristiken gibt, und die Anzahl der verschiedenen Schnittsysteme, wie wir gesehen haben, 51806208 beträgt, so sind jeder Characteristik 104448 Schnittsysteme coordiniert.

In den beiden Figuren 16 und 17 sind die Verzweigungspunkte und Uebergangslinien die stereographische Projection der Ecken und zehn Kanten eines Icosaeders von einer der Ecken aus, und zwar steht die Achse durch diese Ecke senkrecht auf der Ebene und die zehn Kanten, die diese Achsen treffen, sind diejenigen, welche projicirt sind. Es sind also die zwölf Verzweigungspunkte, die elf in den Figuren gegebenen, zusammen mit dem Punkte im Unendlichen, zu welchem die Uebergangslinien von den äusseren fünf Verzweigungspunkten alle hinführen. Es müssen in diese Figuren noch Linien gezogen werden, die jedes Schnittpaar mit irgend einem beliebig angenommenen Punkte verbinden, damit die Fläche einfach zusammenhängend ist. Die Schnitte können so genommen werden, dass sie die Verzweigungspunkte in der Richtung des Uhrzeigers einschliessen, mit Ausnahme der Schnitte B in Figur 17, wo die positive Richtung dem Uhrzeiger entgegengesetzt ist.

Die Transformationsdeterminante, mittels deren die Figur 17 aus Figur 16 entsteht ist folgende (wobei, nachdem diese Transformation ausgeführt ist, dem Schnittsystem die "einfachste" Form des §13 gegeben ist):

$$\left|\begin{array}{ccccc ccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\
 & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right|'$$

Wie §14 zeigen wird, ist die Bedingung die das Schnittsystem erfüllen muss, damit die Characteristik $\binom{11111}{11111}$ mit der Characteristik $\phi_0 . \psi_{2.5+2}$ verbunden ist, die, dass jeder der Schnitte eine ungerade Anzahl Paare von Verzweigungs-

punkten einschliesst. Deshalb kann man andere Schnittsysteme mit derselben Characteristik $\binom{11111}{11111}$ erhalten wenn man anstatt A_i: $A'_i = A_i + B_i$ (wo A'_i die "einfachste" Form von §13 ist) nimmt, und die anderen Schnitte unverändert lässt, oder wenn man statt B_i: $B'_i = A_i + B_i$ (wo B'_i auch die "einfachste" Form annimmt), oder Combinationen solcher Aenderungen setzt. Auch kann jedes der so erhaltenen Schnittsysteme stereographisch aus jeder der Ecken des Icosaeders projicirt werden, und so bekommt man verschiedene Schnittsysteme mit derselben Characteristik.

§10.—*Die neun Familien für* $p = 5$.

Die 496 Characteristiken $\binom{g}{h}$ für $p = 5$ können in neun Familien eingeteilt werden, wie die für $p = 3$ in sechs geteilt sind. Hier sollen Transformationsdeterminanten (cf. §13) gegeben werden, die von irgend einem der 104448 Schnittsysteme mit der Characteristik $\binom{11111}{11111}$ zu den acht Repräsentanten führen:

$$\text{II}, \binom{11100}{11111}; \quad \text{III}, \binom{11100}{11110}; \quad \text{IV}, \binom{11100}{11100}; \quad \text{V}, \binom{10000}{11111};$$

$$\text{VI}, \binom{10000}{11110}; \quad \text{VII}, \binom{10000}{11100}; \quad \text{VIII}, \binom{10000}{11000}; \quad \text{IX}, \binom{10000}{10000}.$$

Die Schnittsysteme welche den anderen 485 Characteristiken angehören können aus dem gegebenen System durch Transformationen abgeleiten werden, welche denfolgenden acht ähnlich sind:

$$\text{II}, \begin{vmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&-1 \\ 0&0&0&0&1&0&0&0&-1&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{vmatrix} \qquad \text{III}', \begin{vmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&-1 \\ 0&0&0&0&1&0&0&0&-1&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&-1&1 \end{vmatrix}'$$

$$
\mathrm{IV}',\ \left|\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\[6pt]
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1
\end{array}\right|'
\qquad
\mathrm{V},\ \left|\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\[6pt]
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right|
$$

$$
\mathrm{VI}',\ \left|\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\[6pt]
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1
\end{array}\right|'
\qquad
\mathrm{VII}',\ \left|\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\[6pt]
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1
\end{array}\right|'
$$

$$
\mathrm{VIII}',\ \left|\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\[6pt]
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1
\end{array}\right|'
\qquad
\mathrm{IX}',\ \left|\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\[6pt]
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1
\end{array}\right|'
$$

III.—Zusammenordnung der transcendentalen und algebraischen Characteristiken.

§11.—*Die beiden Arten Thetacharacteristiken.*

Auf der hyperelliptischen Fläche betrachten wir die gewöhnlichen 2^{2p}, durch die folgende Formel definirten Thetafunctionen:

$$\vartheta_{\binom{g_1, \ldots, g_p}{h_1, \ldots, h_p}}(v_1, \ldots, v_p; \tau) =$$

$$\sum_{n_1}^{+\infty}{}_{-\infty} \ldots \sum_{n_p}^{+\infty}{}_{-\infty} e^{i\pi \sum_{\alpha}^{p}{}_{1} \sum_{\beta}^{p}{}_{1} \left(n_\alpha + \frac{g_\alpha}{2}\right)\left(n_\beta + \frac{g_\beta}{2}\right)\tau_{\alpha\beta} + 2i\pi \sum_{\alpha}^{p}{}_{1} \left(n_\alpha + \frac{g_\alpha}{2}\right)\left(v_\alpha + \frac{h_\alpha}{2}\right)}.$$

Da jedes g und h die Werthe Null oder Eins annehmen kann, so hat man 2^{2p} solcher Functionen, jede mit einem verschiedenen Symbol:

$$\begin{pmatrix} g_1, & g_2, & \ldots, & g_{p-1}, & g_p \\ h_1, & h_2, & \ldots, & h_{p-1}, & h_p \end{pmatrix},$$

und dieses Symbol wird die *transcendentale Characteristik* der Thetafunction genannt. Eine Thetafunction mit einer solchen Characteristik soll in der verkürzten Form $\vartheta_{\binom{g}{h}}$ dargestellt werden.

Aber andererseits sind die 2^{2p} Thetas auch Functionen der $2p + 2$ Verzweigungspunkte. Wenn $f_{2p+2}(z)$ die Form ist, deren Wurzeln die Verzweigungspunkte der Riemann'schen Fläche sind, und diese Form in die Factoren $\phi(z) \, . \, \psi(z)$ zerlegt wird, deren Ordnungen $p + 1 - 2\mu$ respective $p + 1 + 2\mu$ sind, wo μ alle ganzzahligen Werthe zwischen Null und $\frac{1}{2}(p + 1)$ inclusive annehmen kann, so ist das einzelne Theta eine Function der Zerspaltung dieser $(2p + 2)$ Verzweigungspunkte in die beiden Factoren ϕ und ψ und diese Zerspaltungen können auf 2^{2p} verschiedene Weisen vor sich geben. Die Wurzeln von $f(z)$, welche in einem der Factoren liegen, mögen bezeichnet werden durch $\zeta_1, \zeta_2, \ldots, \zeta_m$ die des anderen durch $\eta_1, \eta_2, \ldots, \eta_n$ wo n eine der beiden Zahlen $p + 1 - 2\mu$, respective $p + 1 + 2\mu$, und m die andere ist.

Wenn für irgend ein Theta nur die Angabe der Anzahl der Verzweigungspunkte für jeden Factor $\phi \, . \, \psi$ gewünscht wird, so legt man dem Theta die "Characteristik" μ bei; wenn man aber ganz besonders hervorheben will,

welche Verzweigungspunkte in ϕ beziehungsweise in ψ liegen, so ist die Characteristik der Thetafunction folgende:

$$\phi, \psi \begin{pmatrix} \zeta_1, \zeta_2, \ldots . \zeta_m \\ \eta_1, \eta_2, \ldots . \eta_n \end{pmatrix}.$$

Man nennt dies die algebraische Characteristik der Function und wie wir oben gesehen haben, gibt es 2^{2p} dieser Characteristiken. Die Abkürzung für diese Thetafunction wird sein $\vartheta_{\phi, \psi}$.

Dieselbe Thetafunction wird also nicht nur ein Characteristik $\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right)$ sondern auch eine ϕ, ψ haben und das ϕ, ψ, welches zu einem gegebenen $\vartheta_{\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right)}$ gehören soll, wird dabei von dem gewählten Schnittsystem abhängen.

Schon in Rosenhains* Artikel über die Thetafunctionen ist die Annahme der Thetas als solcher Functionen der Verzweigungspunkte für $p = 2$ eingeführt, und zu diesem Fall hat Weierstrass† die Coordination gegeben; andererseits ist die Coordination für $p = 3$ von Henoch‡ gegeben worden. C. Neumann‖ unterzog die Thetas, welche zu der Zerspaltung gehören, wo μ gleich Null ist und J. Thomae§ die Quotienten der Thetas desselben μ einer näheren Betrachtung. Erst Prym in seinem schon citirten Artikel gibt für jedes Theta eine Characteristik, die nur von den Verzweigungspunkten abhängt. Die jetzt angewandte Bezeichung unterscheidet sich von der der älteren Autoren, ganz besonders von der Pryms, die dadurch nicht symmetrisch ist, dass einer der Verzweigungspunkte im Unendlichen angenommen wird. Und so wird bei den früheren Autoren ein Unterschied in der Characteristik gemacht, jenachdem dieser Punkt unter den Wurzeln von $\phi_{p+1-2\mu}(z)$ oder von $\psi_{p+1+2\mu}(z)$ vorkommt.

§12.—*Die Vorzeichenfactoren an den Schnitten eines gegebenen Systems.*

Es entsteht die Frage, welches $\vartheta_{\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right)}$, bei einem gegebenen Schnittsystem mit einem gegebenen $\vartheta_{\phi, \psi}$ verbunden werden soll. Diese Frage soll nach den in den Mathematischen Annalen, Bd. 32, von den Herren Prof. Klein und Dr. Burk-

* Recueil des mémoires des Savants étrangers, t. XI, Paris, 1851.

† Cf. Königsberger, Crelle's Journal, Band 64 (1864).

‡ Berliner Dissertation (1867).

‖ Abel'sche Integrale (1865), Zwölfte Vorlesung.

§ Crelle's Journal, Band 71 (1870).

hardt gegebenen Principien beantwortet werden. Diese Coordination wird für alle Systeme dadurch festgelegt, dass die Vorzeichen der Factoren, welche zu $\vartheta_{\binom{g}{h}}$ und $\vartheta_{\phi,\,\psi}$ treten, beim Ueberschreiten der Schnitte bestimmt werden; denn diese Vorzeichen bei den $2p$ Schnitten des Systems müssen dieselben sein für dieselben Functionen.

Für $\vartheta_{\binom{g}{h}}$ sind diese Vorzeichen schon bekannt, da die Function den beiden Functionalgleichungen genügt:

$$\vartheta_{\binom{g_1,\,\ldots,\,g_p}{h_1,\,\ldots,\,h_p}}(v_1,\ldots,v_k+1,\ldots,v_p)=(-1)^{g_k}.\vartheta_{\binom{g_1,\,\ldots,\,g_p}{h_1,\,\ldots,\,h_p}}(v_1,\ldots,v_k,\ldots,v_p)$$

$$\vartheta_{\binom{g_1,\,\ldots,\,g_p}{h_1,\,\ldots,\,h_p}}(v_1+\tau_{1k},\ldots,v_a+\tau_{ak},\ldots,v_p+\tau_{pk})$$
$$=(-1)^{h_k}.e^{G(v)}.\vartheta_{\binom{g_1,\,\ldots,\,g_p}{h_1,\,\ldots,\,h_p}}(v_1,\ldots,v_a,\ldots,v_p).$$

Diese beiden Gleichungen zeigen, dass, wenn die oberen Gränzen der Argumente: $v_1,\ldots v_p$ die zu einem gegebenen Systeme gehörenden Schnitte überschreiten:

$$A_1,\quad A_2,\quad A_3,\ldots,\quad Ap;\quad B_1,\quad B_2,\quad B_3,\ldots,\quad B_p$$

die Thetafunction ihrerseits, von anderen Bestandtheilen abgesehen, die Factoren:

$$(-1)^{g_1},\ (-1)^{g_2},\ (-1)^{g_3},\ldots,\ (-1)^{g_p};\ (-1)^{h_1},\ (-1)^{h_2},\ (-1)^{h_3},\ldots,\ (-1)^{h_p}$$

annimmt.

Und so kann wiederum, sobald diese Factoren für eine Thetafunction gegeben sind, ihre transcendentale Characteristik $\begin{pmatrix} g_1\ldots g_p \\ h_1\ldots h_p \end{pmatrix}$ aus diesen Factoren bestimmt werden.

Wie wird es aber mit der Function mit algebraischer Characteristik, $\vartheta_{\phi,\,\psi}$, werden? Im 32. Bande der Mathematischen Annalen ist gezeigt worden, dass, wenn das Argument um eine "Elementarschleife" genommen wird (d. h. eine solche, welche nur zwei Verzweigungspunkte enthält und welche um jeden nur einmal und zwar in positiver Richtung herumgeht) die Function $\vartheta_{\phi,\,\psi}$ die Factoren $(-1)^0$ oder $(-1)^1$ annimmt, je nachdem die beiden eingeschlossenen Verzweigungspunkte in der zugehörigen Zerspaltung $f_{2p+2}(z)=\phi_{p+1-2\mu}(z).\psi_{p+1+2\mu}(z)$ getrennt oder zusammen sind. Die Hinführung des Arguments um eine Schleife, welche um eine gerade Anzahl Verzweigungspunkte und zwar um jede nur einmal und in derselben Richtung (eine sogenannte "directe" Schleife) geht,

kann auf die Hinführung des Arguments um eine Anzahl elementarer Schleifen zurückgeführt werden, nämlich derjenigen in welche die erste Schleife gespaltet werden kann; und $\vartheta_{\phi,\psi}$ wird um die ganze Schleife geführt das Produkt der Vorzeichenfactoren erhalten, welche es um jede einzelne der Elementarschleifen erhält. Dies kann ausgedrückt werden durch den Satz: Um eine directe Schleife, wo die Anzahl der eingeschlossenen Verzweigungspunkte congruent zwei (modulo 4) ist, erhält die $\vartheta_{\phi,\psi}$ function den Factor $(-1)^1$ oder $(-1)^0$, je nachdem eine gerade oder ungerade Anzahl eingeschlossener Verzweigungspunkte in ϕ liegt; wenn aber die Anzahl der eingeschlossenen Verzweigungspunkte congruent Null (modulo 4) ist, so erhält die Function den Factor $(-1)^0$ oder $(-1)^1$, je nachdem eine gerade oder ungerade Anzahl eingeschlossener Verzweigungspunkte in ϕ liegt. Dieses Resultat kann in die einfache Formel gebracht werden: Wenn $2a$ die Anzahl der Verzweigungspunkte in einer directen Schleife und b deren Anzahl darstellt, welche Wurzeln in $\phi(z)$ oder $\psi(z)$ sind, so ist der Vorzeichenfactor dieser Schleife:

$$(-1)^{a+b}.$$

Aber die allgemeinste in sich zurücklaufende Schleife auf der Riemann'schen Fläche besteht aus directen Schleifen und einer Anzahl doppelter Drehungen um gewisse einzelne Verzweigungspunkte, und jede dieser doppelten Drehungen gilt für eine elementare Schleife mit dem Vorzeichenfactor $(-1)^1$. Wenn nun die Anzahl dieser doppelten Drehungen gleich c, und $2a$ die Anzahl der in allen directen Schleifen eingeschlossenen Punkte ist, und b die Anzahl dieser letzteren Punkte in $\phi(z)$ (oder $\psi(z)$), so ist der Vorzeichenfactor für die ganze Schleife:

$$(-1)^{a+b+c}.$$

Oder, da der folgende Satz für die elementaren Schleifen und für die doppelten Drehungen, in welche irgend eine Schleife zerspaltet werden kann, richtig ist, so ist er auch im Allgemeinen richtig: Wenn d die algebraische Summe der einfachen Drehungen in positiver Richtung um die eingeschlossenen ϕ-punkte und e dieselbe Anzahl für die eingeschlossenen ψ-punkte ist, dann ist der Vorzeichenfactor für den ganzen Schnitt:

$$(-1)^{\frac{d-e}{2}}.$$

§13.—*Dasselbe für ein abgeleitetes System.*

Nach einer der oben gegebenen Regeln können die Vorzeichenfactoren um alle Schnitte irgend eines canonischen Schnittsystems bestimmt werden, sobald die Zerspaltung der Verzweigungspunkte zwischen ϕ und ψ festgelegt ist. Bei diesem Punkte entsteht naturgemäss die Frage: welches werden die Vorzeichenfactoren sein, welche an den Schnitten eines anderen Schnittsystems entstehen, das mit dem ersten durch eine gegebene Transformationsdeterminante verbunden ist? Der Factor für das Ueberschreiten eines Schnittes ist der, welchen man erhält, wenn man um den anderen Schnitt des Paares herumgeht; und die Periode an irgend einem Schnitt des neuen Systems wird gegeben durch eine lineare Combination von den Schnitten des alten Systems. Aber man wird sich erinnern, dass bei der Betrachtung des Schnittsystems, welches von einem anderen durch lineare Transformation abgeleitet war, gezeigt ist, dass Linien die sich nur durch eine doppelte Drehung um Verzweigungspunkte unterscheiden, nicht in ihrem Wesen verschieden sind, und es wird gewöhnlich richtiger sein, jeden Schnitt in der einfachsten Form zu betrachten, die er annehmen kann,

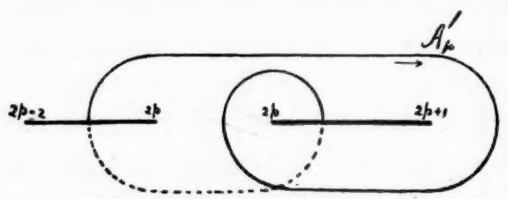

FIG. 18.

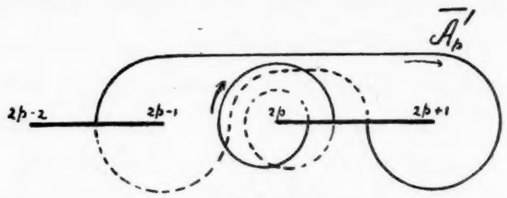

FIG. 19.

d. h. die Form, in welcher er um Paare eingeschlossener Verzweigungspunkte in derselben Richtung herumgeht. Um den einfachsten Fall zu nehmen: in Figur 5 wird der Schnitt $A_p' = A_p - B_p$ die in Figur 18 gegebene Form haben, die von Figur 19 nicht wesentlich verschieden ist. Die Transformation $A_p' = A_p - B_p$ jedoch bezeichnet den Schnitt A_p' und nicht $\overline{A_p'}$, und $\vartheta_{\phi\psi}$ hat um die beiden Schnitte ein verschiedenes Vorzeichen. Es wird sich empfehlen, das folgende Symbol einzuführen: Ein zu der Transformationsdeterminante hinzugefügter Strich soll anzeigen, dass nachdem die angedeutete Transformation ausgeführt ist, die Schnitte in ihrer einfachsten Form genommen werden müssen, d. h. durch die Addition einer ungeraden Anzahl von doppelten Drehungen um Verzweigungspunkte muss der Schnitt so gelegt werden, dass er um Paare eingeschlossener Verzweigungspunkte je in derselben Richtung herumgeht. Wenn

nun irgend ein abgeleitetes Schnittsystem in seiner einfachsten Form betrachtet wird, d. h. wenn die Schnitte in der oben beschriebenen einfachsten Form angenommen sind, so wird die aus einem Schnittpaar zusammengesetzte Schleife einen Vorzeichenfactor geben, der das Produkt von $(-1)^1$ multiplicirt mit dem Produkte der Vorzeichenfactoren der einzelnen Schnitte des Paares ist. Denn die zwei Schnitte eines Paares schliessen immer eine ungerade Anzahl gemeinsamer Punkte ein und deshalb wird die doppelte Drehung um einen dieser Punkte einen Vorzeichenfactor $(-1)^1$ geben, aber diese doppelten Drehungen dürfen, wie wir oben gesagt haben, nicht als in den Schnitt eingeschlossen betrachtet werden. Wenn also f die Anzahl der Schnittpaare in der linearen Combination ist, so wird, da irgend eine lineare Combination von Schnitten aus einer Combination von Schnittpaaren und aus einer Combination von einzelnen Schnitten besteht, die in ihrer einfachsten Form genommene aus einer linearen Combination von den Schnitten irgend eines canonischen Systems zusammengesetzte Schleife einen Vorzeichenfactor haben, der das Produkt von $(-1)^f$ multiplicirt mit dem Produkte der Vorzeichenfactoren aller der Schnitte der linearen Combinationen ist. Dies kann auch so ausgedrückt werden: Wenn die Transformationsdeterminante die in §4 gegebene Form hat und Q_λ und $Q_{\lambda\pm p}$ die beiden Schnitte eines Paares und f_λ der Exponent des Vorzeichenfactors des Schnittes Q_λ des ursprünglichen Schnittsystems und f'_κ der des in seiner einfachsten Form genommenen Schnittes Q'_κ des abgeleiteten Systems sind, so wird der abgeleitete Schnitt Q'_κ einen Vorzeichenfactor haben, dessen Exponent ist:

$$f'_\kappa = \sum_1^{2p} c_{\kappa\lambda} f_\lambda + \sum_1^{p}{}^{\lambda} f_\lambda f_{\lambda\pm p}.$$

§14.—*Zusammenordnung für das Weierstrass'sche Schnittsystem.*

Wir sind jetzt in der Lage, die Coordination der $\vartheta_{\binom{g}{h}}$- und $\vartheta_{\phi,\psi}$-Functionen und zwar erstens für die Weierstrass'schen Perioden zu bestimmen. Wenn die $2p+2$ Verzweigungspunkte in der folgenden Reihenfolge genommen $2p+2,\ 1,\ 2, \ldots,\ 2k-1,\ 2k,\ 2l-1,\ 2l, \ldots,\ 2p-1,\ 2p,\ 2p+1$ sind, so werden für das Weierstrass'sche Schnittsystem (siehe Figur 5) die Schnitte

$$B_1,\quad B_2,\ \ldots,\quad B_k,\qquad B_l,\qquad \ldots,\qquad B_p,$$

beziehungsweise die Paare

$$1,\ 2;\ 3,\ 4;\ \ldots;\ 2k-1,\ 2k;\ 2l-1,\ 2l;\ \ldots;\ 2p-1,\ 2p$$

einschliessen und diesen Schnitten entsprechen in der transcendentalen Characteristik die Elemente

$$g_1,\quad g_2,\quad \ldots,\quad g_k,\quad g_l,\quad \ldots,\quad g_p.$$

Aber da diese Schnitte alle elementare Schleifen sind, so wird die algebraische Characteristik $2k-1,\ 2k$ zusammen enthalten, wenn $g_k=1$ ist, und getrennt, wenn $g_k=0$ ist, und dasselbe gilt für die anderen Punktpaare in den Schnitten B. Die Schnitte A sind in jedem Falle directe Schleifen, von denen jede zwei Verzweigungspunkte mehr einschliesst als die vorhergehende. Der Schnitt A_p (respective A_1) ist eine elementare Schleife um die Punkte $2p,\ 2p+1$ (respective $1,\ 2$) und daher ist h_p (respective h_1) $=1$ oder $=0$, jenachdem die Punkte $2p,\ 2p+1$ (respective $1,\ 2$) zusammen oder getrennt in der algebraischen Characteristik liegen. Der Schnitt A_k besteht aus einer elementaren Schleife um $2k-1,\ 2k$ und der directen Schleife A_l; wenn nur $2k-1,\ 2k$ getrennt in der Characteristik $\begin{pmatrix}\zeta_1, \zeta_2, \ldots, \zeta_m\\ \eta_1, \eta_2, \ldots, \eta_n\end{pmatrix}$ liegen, so wird, da der Vorzeichenfactor dieser elementaren Schleife $(-1)^0$ ist, der Schnitt A_k denselben Vorzeichenfactor haben, wie der Schnitt A_l; aber wenn im Gegentheil $2k-1,\ 2k$ zusammen in der algebraischen Characteristik liegen, so wird der Vorzeichenfactor dieser elementaren Schleife $(-1)^1$ sein, und der Schnitt A_k wird einen Vorzeichenfactor haben, der verschieden von dem zu Schnitte A_l gehörigen ist. Alles dieses kann ausgedrückt werden durch eine

Tabelle welche die Elemente der transcendentalen und algebraischen Characteristiken für das Weierstrass'sche System verbindet:

Verzweigungspunkte	$1,\ 2$	$2k-1,\ 2k$	$2p-1,\ 2p$	$2p+2,\ 1$	$2k,\ 2l-1$	$2p,\ 2p+1$
zusammen	$g_1=1$	$g_k=1$	$g_p=1$	$h_1=1$	$h_k+1\equiv h_l \pmod 2$	$h_p=1$
getrennt	$g_1=0$	$g_k=0$	$g_p=0$	$h_1=0$	$h_k=h_l$	$h_p=0$

Aus dieser Tabelle kann man sofort die andere Characteristik ablesen, wenn die eine gegeben ist.

Beispielsweise sind die folgenden Characteristiken coordinirt:

Für $p \equiv 1$ (mod. 2):

$$\begin{pmatrix} 1111 \ldots . 111 \\ 1010 \ldots . 101 \end{pmatrix} \ldots . \phi_0 \psi_{2p+2} \begin{pmatrix} 2p+2, 1, 2, \ldots ., 2p+1 \\ \ldots . \end{pmatrix},$$

für $p \equiv 0$ (mod. 2):

$$\begin{pmatrix} 111 \ldots . 111 \\ 010 \ldots . 101 \end{pmatrix} \ldots . \phi_1 \psi_{2p+1} \begin{pmatrix} 1, 2, \ldots ., 2p+1 \\ 2p+2 \end{pmatrix},$$

für jedes p:

$$\begin{pmatrix} g_1, \ldots . g_{p-1} 0 \\ h_1, \ldots . h_{p-1} 0 \end{pmatrix} \ldots \begin{pmatrix} \zeta_1, \ldots . \zeta_{m-2}, 2p-1, 2p+1 \\ \eta_1, \ldots . \eta_{n-1}, 2p \end{pmatrix},$$

$$\begin{pmatrix} g_1, \ldots . g_{p-1} 1 \\ h_1, \ldots . h_{p-1} 0 \end{pmatrix} \ldots \begin{pmatrix} \zeta_1, \ldots ., \zeta_{m-1}, 2p+1 \\ \eta_1, \ldots ., \eta_{n-2}, 2p-1, 2p \end{pmatrix},$$

$$\begin{pmatrix} g_1 \ldots . g_{p-1} 0 \\ h_1 \ldots . h_{p-1} 1 \end{pmatrix} \ldots \begin{pmatrix} \zeta_1 \ldots . \zeta_{m-2}, 2p, 2p+1 \\ \eta_1 \ldots . \eta_{n-1}, 2p-1 \end{pmatrix},$$

$$\begin{pmatrix} g_1 \ldots . g_{p-1} 1 \\ h_1 \ldots . h_{p-1} 1 \end{pmatrix} \ldots \begin{pmatrix} \zeta_1 \ldots . \zeta_{m-3}, 2p-1, 2p, 2p+1 \\ \eta_1 \ldots . \eta_n \end{pmatrix},$$

$$\begin{pmatrix} g_1, \ldots ., g_{k-1}, 0, g_{k+1}, \ldots ., g_p \\ h_1, \ldots ., h_{k-1}, h_k, h_{k+1}, \ldots ., h_p \end{pmatrix} \ldots \begin{pmatrix} \zeta_1, \ldots ., 2k-1, \ldots . \zeta_m \\ \eta_1, \ldots ., 2k, \quad \ldots . \eta_n \end{pmatrix},$$

$$\begin{pmatrix} g_1, \ldots ., 1, \ 1, \ldots . g_p \\ h_1, \ldots ., h_k, h_l, \ldots . h_p \end{pmatrix} \ldots \begin{pmatrix} \zeta_1, \ldots ., 2k-1, 2k, \ldots ., \zeta_m \\ \eta_1, \ldots ., 2l-1, 2l, \ldots ., \eta_n \end{pmatrix}.$$

$$(h_k = h_l)$$

Aus der Zusammenordnung der Characteristiken für $p \equiv 1$ (mod. 2) sieht man dass zu $\phi_0 \psi_{2p+2}$ für $p=3$ eine gerade Characteristik (§7), für $p=5$ dagegen eine ungerade Characteristik (§9) coordinirt ist.

§15.—*Zusammenordnung für irgend ein Schnittsystem.*

Für jedes beliebige Schnittsystem können die Factoren an den verschiedenen Schnitten, wenn die Characteristik ϕ, $\psi \begin{pmatrix} \zeta_1, \zeta_2, \ldots ., \zeta_m \\ \eta_1, \eta_2, \ldots ., \eta_n \end{pmatrix}$ gegeben ist, nach den in §12 gegebenen Regeln gefunden werden, und es ist deshalb das correspondirende $\begin{pmatrix} g_1, g_2, \ldots ., g_p \\ h_1, h_2, \ldots ., h_p \end{pmatrix}$ sofort bekannt. Auch wenn die transcendentale Characteristik gegeben ist, kann die algebraische Characteristik in einigen Fällen durch Umkehrung derselben Regeln abgeleitet werden; z. B. corre-

spondirt in dem Schnittsystem der Figur 10 mit der transcendentalen $\begin{pmatrix} 000 \\ 000 \end{pmatrix}$ die algebraische $\phi_0 . \psi_8$; denn jeder Schnitt enthält vier Verzweigungspunkte, und da hier an jedem Schnitt der Vorzeichenfactor $(-1)^0$ ist, so liegen die vier eingeschlossen Verzweigungspunkte in zwei Paaren in ϕ und ψ und da es sechs solcher Schnitte gibt, wie man leicht ersehen kann, so müssen die acht Punkte zusammen liegen.

Aber auch ganz allgemein kann die algebraische Characteristik gefunden werden, wenn die transcendentale Characteristik gegeben ist. Wenn nämlich die Abel'sche Substitution bekannt ist, durch die aus diesem gegebenen Schnittsystem das Weierstrass'sche System abgeleitet wird (und diese Substitution wird leicht gefunden wenn man einfach die beiden Systeme übereinander legt und beobachtet, wie die Schnitte des einen Systems die des anderen Systems kreuzen), so wird nach den in §13 gegebenen Regeln die mit dem gegebenen $\begin{pmatrix} g_1 g_2 \dots g_p \\ h_1 h_2 \dots h_p \end{pmatrix}$ correspondirende transcendentale Characteristik des Weierstrass'schen Systems gefunden, und aus dieser letzteren Characteristik kann die algebraische Characteristik nach den in §14 gegebenen Regeln berechnet werden.

GÖTTINGEN, *den* 8. *Juli* 1892.

INHALT.

On the Determination of Groups whose Order is a Power of a Prime.

By J. W. A. Young.

1. Cayley* has called attention to an important desideratum in the theory of groups, viz. the determination of all groups of given order n,—two groups of the same order being considered identical when the laws of combination of their elements coincide.† He had previously, in papers on the theory of groups as depending on the symbolic equation $\vartheta^n = 1$,‡ determined the groups in the first case of any difficulty, viz. $n = 8$.

Kempe‖ has given all groups of order 1 to 12, and subsequently Cayley,§ correcting some errors made by Kempe, gave the groups of the same orders, together with a method of graphic representation. So much by way of enumeration from order to order. On the other hand, Netto¶ gives all groups of order p^2 and pq, p and q both being prime numbers. Finally, Kronecker** has completely determined all commutative groups of any finite order.

2. In the following paper, groups whose order is a power of a prime are considered. In the first paragraph the group-theoretic foundations are laid and a method is reached of classifying groups whose order is a power of a prime in distinct "types". In the second paragraph the groups of order p^3 and those

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* Am. J. Math., Vol. I, p. 50.

† *i. e.* all holoedrically isomorphic groups are regarded as identical.

‡ Phil. Mag., Vol. VII, 1854, pp. 40, 408; Vol. XVIII, 1859, p. 34.

‖ Phil. Trans., Vol. CLXXVII, 1886, pp. 37-43.

§ Am. J. Math., Vol. XI, p. 139.

¶ Substitutionentheorie, pp. 133-37. Netto's results are, on the face of it, proved only for groups of substitutions, but they hold generally, since every group can be expressed as a group of substitutions. *Vid.* Cayley, Am. J. Math., Vol. I, p. 52; Dyck, Math. Annalen, Vol. XXII, p. 84.

** Berl. Monatsberichte, 1870, p. 881. *Cf.* Schering, Gött. Abhandlungen, July 11, 1868, Vol. XIV.

of order p^4 are completely determined, while the third paragraph is devoted to further consideration of the groups of order 2^4. The fourth paragraph consists of the treatment of those general types in which I have succeeded in finding the distinct groups. The division into types, given in the first paragraph, makes it possible to write out the generating relations of any group of order p^x, and readily to reduce the question of the identity of two groups to that of "equivalence" of two sets of integers. The method is sufficiently explained and illustrated in the types treated in the second and the fourth paragraph.

3. In the composition of the paper, it has been my aim to presuppose no knowledge of the theory of groups on the part of the reader, and throughout the paper (the notes excepted) the terms used have been defined and the theorems used proved, or else reference has been made to at least one work where such definition or proof may be found. The third paragraph assumes some acquaintance with substitution-groups and with Cayley's method of graphic representation of groups.

I take this opportunity to express my obligation and warmest thanks to Dr. Oskar Bolza, who, throughout the entire course of the investigation, very kindly and materially assisted me with suggestions and counsel, as well as to Professor W. E. Story for valuable criticisms.

§1.

4. The objects of consideration are elements $A_1A_2 \ldots .$, $B_1B_2 \ldots .$, $\ldots .$ having the following defining properties:*

1) *Multiplication, i. e. a mode of composition, is unambiguously defined, and is associative, but not, in general, commutative.*

2) *Division, i. e. a mode of decomposition, is unambiguously defined, viz. by the property that from* $AB = AC$, *as well as from* $BA = CA$, *shall always follow* $B = C$.

3) *Abstraction is made of all properties of the elements in so far as they are not consequences of these just mentioned; i. e., abstraction is made of the content of the*

* Kronecker, Berl. Monatsberichte, 1870, p. 882; Weber, Math. Annalen, Vol. XX, p. 302; Frobenius, Crelle's Journal, Vol. C, pp. 179–80; Perott, Am. J. Math., Vol. XI, pp. 99–101; Hölder, Math. Annalen, Vol. XXXIV, p. 29.

symbol. Two elements B and C are identical when $AB = AC$, as well as $BA = CA$, for every element A.

5. In consequence of 1) the terminology of multiplication is employed and the element AB, resulting from the combination of A and B, is called the *product* of the *factors* A and B. (It is not necessary to the purposes of this paper to make a convention as to which element shall be considered multiplier and which multiplicand.) The element $AA = A^2$, resulting from the combination of the element A with itself, is styled the second *power* of A, and, in general, $A.A.A....(\alpha \text{ times}) = A^\alpha$, the α^{th} *power* of A.

6. A *group* is an aggregate of elements such that every product of two elements of the aggregate is also contained in the aggregate. The *order* of a group is the number of distinct elements it contains. From the assumptions and definitions above it follows easily:*

a.) That in every group of finite order there exists one, and only one, element A_0 such that for every element A of the group $A_0A = AA_0 = A$; A_0 is called the *principal* element of the group and denoted by the symbol 1.

b.) That, if A_i be any element of the group, there always exists in the group one, and only one, element A_i' such that $A_iA_i' = A_i'A_i = A_0 = 1$. A_i' is called the *inverse* of A_i and is written A_i^{-1}.

7. If the group contains the elements $A_1, A_2, \ldots . A_m$, every expression of the form

$$A_1^{\alpha_{11}} A_2^{\alpha_{12}} \ldots . A_m^{\alpha_{1m}} A_1^{\alpha_{21}} A_2^{\alpha_{22}} \ldots . A_m^{\alpha_{2m}} \ldots . A_1^{\alpha_{l1}} A_2^{\alpha_{l2}} \ldots . A_m^{\alpha_{lm}}$$

is also an element of the group. If it happens that *every* element of the group can be expressed in terms of the elements $A_1A_2 \ldots . A_m$ in the form just given, $A_1A_2 \ldots . A_m$ are called a *system of generating elements* of the group. A group is completely defined when the laws of combination of a system of generating elements are given, and is said to be *generated by* these elements. The laws of combination are always expressible as equalities between two products of the form given above, or, denoting such a product by $F(A_1A_2 \ldots . A_m)$ as

$$F_1(A_1A_2 \ldots . A_m) = F_2(A_1A_2 \ldots . A_m).$$

By multiplying by the inverses of the elements on the right, these relations

*Kronecker, Weber, *l. c.*

can always be brought into the form $F(A_1A_2 \dots . A_m) = 1$, or, for brevity, $F = 1$.

8. From the given relations others usually follow as consequences. In the determination of groups of a given order it is of importance that all the consequences of the relations given be *explicitly* considered. The consequences of given relations are obtained by the following method:*

Let $F = 1$ be a relation between the generating elements $A_1A_2 \dots . A_m$. If

$$F = A_1^{a_{11}} A_2^{a_{12}} \dots . A_m^{a_{1m}} A_1^{a_{21}} A_2^{a_{22}} \dots . A_m^{a_{2m}} \dots A_1^{a_{r1}} A_2^{a_{r2}} \dots . A_m^{a_{rm}} = 1,$$

it follows, multiplying this equality by itself, that

$$A_1^{a_{11}} \dots . A_m^{a_{1m}} \dots . A_1^{a_{r1}} \dots . A_m^{a_{rm}} . A_1^{a_{11}} \dots . A_m^{a_{1m}} \dots . A_1^{a_{r1}} \dots . A_m^{a_{rm}} = 1,$$

which we may write $F^2 = 1$. So, in general, it follows from the relation $F = 1$, that also $F^a = 1$, for all positive integral values of α. But there may be still other consequences of $F = 1$, viz., if $F = 1$ be transformed† by each of the generating elements there follow the relations

$$A_i^{-1} F A_i = 1. \qquad i = 1, 2 \dots . m.$$

These relations may all be identical with $F = 1$, (as in the case that the elements $A_1, A_2, \dots . A_m$ are all commutative with one another). In such a case, the only consequences of the relation $F = 1$ are its own powers. If any of the relations found by transformation are distinct from $F = 1$, they must be transformed again by each of the generating elements, and so on until a system of relations is reached such that when any relation of the system is transformed by any generating element of the group, the resulting relation is already contained in the system. This system, together with all the products and powers of relations contained in it, is a complete system of the consequences of the relation $F = 1$. If there exist other relations, $F_2 = 1$, $F_3 = 1, \dots . F_l = 1$, they must be treated similarly. All products of relations of the different systems so found are also consequences. But nothing new will be reached by transforming relations like

$$F_1 F_2 = 1,$$

for $A_i^{-1} F_1 F_2 A_i = A_i^{-1} F_1 A_i . A_i^{-1} F_2 A_i$, and this is a product of a consequence of F_1 and a consequence of F_2.

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* Dyck, Math. Annalen, Vol. XX, p. 12.

† An element B is said to be "transformed" by A when the product $A^{-1}BA$ is formed, and the element $A^{-1}BA$ is called "the transformed" of B by A.

If, therefore, a group be generated by a system of elements $A_1 A_2 \ldots . A_m$, satisfying the relations $F_1 = 1$, $F_2 = 1, \ldots . F_l = 1$, a complete system of the relations consequent upon these is obtained by transforming each relation, as well as all the new relations so resulting, by all the generating elements of the group; adding all the new relations so found to the original system; and finally, when nothing new arises by transformation, including all the different products and powers of the relations hitherto known.

9. A system of generating elements may always be found containing any given element of the group, but if the group contains an element different from unity, there will be systems of generating elements that do not contain *all* the elements of the group. The same group may therefore be variously defined by different choices of the generating elements and consequent variation in the relations between them.

To find all distinct groups of a given order we consider first all possible systems of relations between elements which will generate a group of the given order; and secondly, determine which of the groups so generated are distinct. That two groups of order N, defined by two systems of relations, may be identical, it is necessary and sufficient that each group contain elements which satisfy all the generating relations of the other group, but no relation independent of these. For under these conditions each group is contained in the other, i. e. the groups are identical.*

10. *Groups of order p^κ; p a prime.*

Sylow† has proved the following theorem:

If the order of a group is p^κ, p being a prime, any element A of the group is represented once and only once by the expression

$$A = A_0^i A_1^j A_2^l \ldots . A_{\kappa-1}^m,$$

when $i, j, l, \ldots . m = 0, 1 \ldots . p-1$, independently, and where $A_0, A_1, \ldots ., A_{\kappa-1}$ are elements of the group satisfying the following relations:

$$\begin{aligned} A_0^p &= 1 \\ A_1^p &= A_0^a \\ A_2^p &= A_0^b A_1^c \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{\kappa-1}^p &= A_0^r A_1^s \ldots . A_{\kappa-2}^t, \end{aligned}$$

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*Poincaré, Acta Math., Vol. I, p. 10.

†Math. Annalen, Vol. V, p. 588.

and also (A *being any element of the group*)

$$
\begin{aligned}
A^{-1}A_0A &= A_0 \\
A^{-1}A_1A &= A_0^{\alpha}A_1 \\
A^{-1}A_2A &= A_0^{\beta}A_1^{\gamma}A_2 \\
&\cdots\cdots\cdots\cdots \\
A^{-1}A_{\kappa-1}A &= A_0^{\pi}A_1^{\rho}A_2^{\sigma}\cdots\cdots A_{\kappa-2}^{\omega}A_{\kappa-1}.
\end{aligned}
$$

The exponents $a, b, c \ldots\ldots t, \alpha, \beta, \gamma, \ldots\ldots \omega$, *may take independently the values* $0, 1, \ldots\ldots p-1$.

This theorem assures that, when the exponents take successively all possible sets of values, all groups of the order are represented at least once, but it remains to determine which of the groups so represented are distinct; or, in other words, given two sets of exponents in Sylow's formulæ, to decide whether or not the corresponding groups are identical. Instead, however, of thus starting from Sylow's theorem, it is more convenient to introduce a modified classification which makes, from the very beginning, a considerable restriction in the number of possibilities to be considered. This classification is based upon the following theorem, which is the foundation of Sylow's theorem just cited, and is proved by him in that connection:

Every group whose order is a power of a prime contains at least one element which is different from unity and commutative with every element of the group.

11. It follows immediately from this theorem* that every group G, of order p^{κ}, contains a subgroup $\Gamma^{(1)}$, of order p^{κ_1}, $\kappa_1 > 0$, consisting of the totality of all the elements of G which are commutative with every element of G. The two groups G and $\Gamma^{(1)}$ suffice to define, as follows, a new group, called the *quotient* of G by $\Gamma^{(1)}$.†

The elements of G can be arranged in the following table, in which the subgroup $\Gamma^{(1)}$ constitutes the first row:

$$
\begin{array}{llll}
1, & A_2, & A_3 & \ldots\ldots A_{p^{\kappa_1}} \\
B_2, & A_2B_2, & A_3B_2 & \ldots\ldots A_{p^{\kappa_1}}B_2 \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
B_{p^{\kappa-\kappa_1}}, & A_2B_{p^{\kappa-\kappa_1}}, & A_3B_{p^{\kappa-\kappa_1}} & \ldots\ldots A_{p^{\kappa_1}}B_{p^{\kappa-\kappa_1}}.
\end{array}
$$

We form the product of any two elements of the table:

$$A_{\alpha}B_{\beta}A_{\alpha'}B_{\beta'} = A_{\alpha}A_{\alpha'}B_{\beta}B_{\beta'} = A_{\alpha''}B_{\beta}B_{\beta'}.$$

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* Netto, Substitutionentheorie, p. 49. † Hölder, Math. Annalen, Vol. XXXIV, pp. 31-33.

The *row* of $A_{\alpha''}B_\beta B_{\beta'}$ is independent of $A_{\alpha''}$, which occurs in every row, and is definitely determined by B_β and $B_{\beta'}$; i. e. from any two rows of the table, taken in a given order, there follows definitely a third. The rows of the table may therefore be considered as elements; they form a group $G^{(1)}$ of order $p^{\kappa-\kappa_1}$. Hölder calls the group $G^{(1)}$, thus determined by G and $\Gamma^{(1)}$, the *quotient* of G by $\Gamma^{(1)}$, and denotes the relation by the equation

$$G^{(1)} = G \,|\, \Gamma^{(1)}.*$$

Analogously,† $G^{(1)}$ and $\Gamma^{(1)}$ are called the factors of G.

12. Since the group $G^{(1)}$ is independent of the A's, its laws of combination are those of the B's when the factors in the A's are disregarded. The laws of combination of the elements of $G^{(1)}$ are thus unambiguously deduced from the laws of G, by placing therein all the A's equal to unity. To pass conversely from the laws of combination of $G^{(1)}$ to those of G, a factor in the A's must be introduced, and the process is not unambiguous.

13. To the group $G^{(1)}$, of order $p^{\kappa-\kappa_1}$, the previous reasoning can be applied, and thus by successive steps we pass unambiguously from one quotient group to another, until finally a group $G^{(\mu)} = 1$ is reached.

In this process the following groups have been defined:

	ORDER.		ORDER.
G	p^{κ}		
$\Gamma^{(1)}$	p^{κ_1}	$G^{(1)} = G \,\vert\, \Gamma^{(1)}$	$p^{\kappa-\kappa_1}$
$\Gamma^{(2)}$	p^{κ_2}	$G^{(2)} = G^{(1)} \,\vert\, \Gamma^{(2)}$	$p^{\kappa-\kappa_1-\kappa_2}$
....	....	....	....
....	....	....	....
$\Gamma^{(\mu-1)}$	$p^{\kappa_{\mu-1}}$	$G^{(\mu-1)} = G^{(\mu-2)} \,\vert\, \Gamma^{(\mu-1)}$	$p^{\kappa-\kappa_1-\kappa_2-\dots-\kappa_{\mu-1}}$
$\Gamma^{(\mu)}$	p^{κ_μ}	$G^{(\mu)} = G^{(\mu-1)} \,\vert\, \Gamma^{(\mu)} = 1$	$p^{\kappa-\sum_{i=1}^{\mu}\kappa} = 1$

If two groups, G_1 and G_2, are identical, their subgroups $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$, unambiguously defined as above, must be identical, and the quotient groups $G_1^{(1)}$ and $G_2^{(1)}$ must also be identical. The converse does not necessarily hold.

* *Any* self-conjugate subgroup is sufficient for the determination of a quotient group. The group $G^{(1)}$ is meriedrically isomorphic with G.

† Hölder, Math. Annalen, Vol. XXXIV, p. 32; Dyck, Math. Annalen, Vol. XX, p. 14.

Applying the same reasoning to $G^{(1)}$, $G^{(2)} \ldots .$, we find that a necessary, though not sufficient, condition for the identity of G_1 and G_2 is that the groups $\Gamma_1^{(i)}$ and $\Gamma_2^{(i)}$, $i = 1, 2 \ldots . \mu$, coincide respectively throughout.*

14. The groups $\Gamma^{(i)}$ are all commutative groups;† to them we apply the following theorem:‡

If the elements of a group of order n are all commutative with one another, there can always be found in the group a system of elements $a_1, a_2, \ldots . a_m$, satisfying the relations

$$a_1^{r_1} = 1,\ a_2^{r_2} = 1, \ldots . a_m^{r} = 1,\ r_1 r_2 \ldots . r_m = n,$$

r_i divisible by r_{i+1}, such that the expression

$$a_1^{h_1} a_2^{h_2} \ldots . a_m^{h_m}$$

represents each element of the group once, and only once, when $h_i = 1, 2, \ldots . r_i$, independently, $(i = 1, 2, \ldots . m)$.

Every set of r's satisfying the above conditions gives rise to a commutative group of order n, when used as exponents in the above relations. All groups are distinct in which the r's do not coincide throughout.

15. By the theorem just cited, the group $\Gamma^{(i)}$, of order p^{κ_i}, is generated by elements satisfying the relations

$$a_1^{p^{\kappa_{i1}}} = 1,\ a_2^{p^{\kappa_{i2}}} = 1, \ldots . a_{l_i}^{p^{\kappa_{il_i}}} = 1,$$
$$a_h a_j = a_j a_h, \quad h, j = 1, 2 \ldots . l_i \text{ independently},$$
$$\kappa_{i1} + \kappa_{i2} + \ldots . + \kappa_{il_i} = \kappa_i .$$

The group $\Gamma^{(i)}$ is completely defined by the numbers $\kappa_{i1}, \kappa_{i2} \ldots . \kappa_{il_i}$ and the fact that it is a commutative group.

Since G_1 and G_2 cannot be identical unless the groups $\Gamma_1^{(i)}$ and $\Gamma_2^{(i)}$ coincide, the numbers κ_{ij} must coincide respectively throughout, if the groups are identical. This agreement in the values of the numbers κ_{ij} is a necessary, though not

* Cf. Hölder, Math. Annalen, Vol. XXXIV, pp. 32–38.

† For brevity we say "commutative groups," instead of "groups whose elements are all commutative with one another."

‡ Kronecker, Berl. Monatsberichte, 1870, p. 881; Schering, Gött. Abhandlungen, 1868, Vol. XIII; Netto, Substitutionentheorie, pp. 144–47; Weber, Math. Annalen, Vol. XX, pp. 304–7; Frobenius u. Stickelberger, Crelle's Journal, Vol. LXXXVI, pp. 224–36. See also Perott, Am. J. Math., Vol. XI, pp. 99–138; Vol. XIII, p. 235.

sufficient, condition for the identity of two groups. It affords thus a basis of classification. We shall say that all the groups leading to the same set of values $\varkappa_{ij}$ constitute a *type*, and denote the type by the symbol

$$(\varkappa_{11}\varkappa_{12}\ldots\ldots\varkappa_{1l_1})(\varkappa_{21}\varkappa_{22}\ldots\ldots\varkappa_{2l_2})\ldots\ldots(\varkappa_{j1}\varkappa_{j2}\ldots\ldots\varkappa_{jl_j}).$$

A type is of as many *steps* as there are parentheses in its symbol. The commutative groups are groups of one step. Since groups belonging to different types cannot be identical, it remains to determine all the different groups contained in each possible type.

16. A group which can be generated by a single element is called a *cyclic* group.

In a group of more than one step, the last group $\Gamma^{(\mu)}$ cannot be a cyclic group. For convenience, we consider a group of only two steps (such a group must always occur, viz. $G^{(\mu-2)}$) and use the notation of No. 13. To say that $\Gamma^{(2)} = G^{(1)}$ is a cyclic group, is to say that, considering the rows of the table in No. 10 as elements, any element can be expressed as a power of one properly chosen, and that hence the elements B, on which alone the group $G^{(1)}$ depends, must also be powers of one of them.

The group G is then generated by $\Gamma^{(1)}$ and a single element B satisfying the relation $B^{p^{\varkappa}-\varkappa_1} = A^m$, ($A$ being some element of $\Gamma^{(1)}$). Since all the elements of $\Gamma^{(1)}$ are commutative with every element of the whole group, and there is only one generating element other than those of $\Gamma^{(1)}$, it follows that the group G is commutative, contrary to the hypothesis that it is of two steps. It is, therefore, not possible that the last group $\Gamma^{(\mu)}$ be a cyclic group. In other words, *in a group of more than one step, the last parenthesis of the type symbol must contain more than one integer.*

§2.

17. *Groups of order p^2.*

The formally possible types are

$$(11)$$
$$(2)$$
$$(1)(1)$$

Of these the first two are commutative groups, and the last type is non-existent, according to the result of No. 16.

Permanent notation. All elements of G which are commutative with every element of G, shall be denoted by the letter a (with the necessary indices) and its products and powers, and the letter a shall be restricted to such elements.

18. *Groups of order p^3.*

Types:

$$\left.\begin{array}{l}(111)\\(21)\\(3)\end{array}\right\}\text{ commutative groups.}$$
$$(1)(11).$$

Type $(1)(11)$.

The type symbol means that $\Gamma^{(1)}$ is generated by an element a satisfying the relation
$$a^p = 1,$$
and that $G^{(1)} = \Gamma^{(2)}$ is generated by elements $a_1^{(1)}$, $a_2^{(1)}$ satisfying the relations
$$\begin{aligned} a_1^{(1)} &= 1 \\ a_2^{(1)^p} &= 1 \\ a_1^{(1)}a_2^{(1)} &= a_2^{(1)}a_1^{(1)}. \end{aligned}$$

The table of G is (cf. No. 11)

$$\begin{array}{llll l} & & & & G^{(1)} \\ 1, & a, & a^2, \ldots . a^{p-1}, & & 1 \\ b_2, & ab_2, & a^2b_2, \ldots . a^{p-1}b_2, & & a_1^{(1)} \\ b_3, & ab_3, & a^2b_3, \ldots . a^{p-1}b_3, & & a_1^{(1)^2} \\ \cdots & \cdots & \cdots & & \vdots \\ b_{p^2}, & ab_{p^2}, & a^2b_{p^2}, \ldots . a^{p-1}b_{p^2}. & & a_1^{(1)^{p-1}} . a_2^{(1)^{p-1}} \end{array}$$

Since the elements of $G^{(1)}$ are the rows of G, every element of the row corresponding to $a_1^{(1)^2}$ must be, to within a factor in a, the square of some element in the row corresponding to $a_1^{(1)}$, i. e., with perhaps a change in the order of the elements of the row beginning with b_3, we may write $b_3 = b_2^2$. Similarly it follows generally that all the b's are powers and products of two of them (call them b_1 and b_2) corresponding to the same powers and products of $a_1^{(1)}$ and $a_2^{(1)}$.

19. To pass from the relations of $G^{(1)}$ given above to those of G deduced therefrom, $a_1^{(1)}$ and $a_2^{(1)}$ must be replaced respectively by b_1 and b_2, and an arbitrary

factor in a introduced (cf. No. 12). Doing so, it is seen that G is generated by elements a, b_1, b_2, satisfying the relations

$$\begin{aligned} a^p &= 1 \\ b_1^p &= a^{m_1} \\ b_2^p &= a^{m_2} \\ b_2 b_1 &= a^n b_1 b_2 \\ a b_i &= b_i a, \ i = 1, 2. \end{aligned}$$

From these relations follow readily

Formulæ of multiplication:

$$\begin{aligned} b_2^{\beta_2} b_1^{\beta_1} &= a^{\beta_1 \beta_2 n} b_1^{\beta_1} b_2^{\beta_2} \\ (a^{\alpha} b_1^{\beta_1} b_2^{\beta_2})^q &= a^{q\alpha + \frac{q(q-1)}{2} \beta_1 \beta_2 n} b_1^{q\beta_1} b_2^{q\beta_2}. \end{aligned}$$

If $n \equiv 0 \pmod{p}$, then $b_2 b_1 = b_1 b_2$ and the group is commutative, and not of the type in question. Hence $n \not\equiv 0 \pmod{p}$.

If the given relations are transformed by all the generating elements no new relations are found. The only relations consequent upon the given relations are, therefore, their products and powers.

20. If
$$\begin{aligned} m_1 &= 0, 1, 2, \ldots . p - 1 \\ m_2 &= 0, 1, 2, \ldots . p - 1 \\ n &= 1, 2 \ldots\ldots . p - 1, \text{ independently,} \end{aligned}$$

all the groups of the type are generated, and every such set of values used as exponents in the generating relations leads to a group of the type. For the relations suffice to reduce any expression of the form

$$a_1^{\alpha_1} b_1^{\beta_{11}} b_2^{\beta_{12}} a_1^{\alpha_2} b_1^{\beta_{21}} b_2^{\beta_{22}} \ldots . a_1^{\alpha_2} b_1^{\beta_{21}} b_2^{\beta_{22}}$$

to the form $a_1^{\alpha} b_1^{\beta_1} b_2^{\beta_2}$, $(\alpha, \beta_1, \beta_2 < p)$. Therefore every element of the group is given at least once by the expression just written if $\alpha, \beta_1, \beta_2 = 0, 1, 2 \ldots . p - 1$, independently. The group is of order p^3 if all these elements are distinct. If two of the elements so found were equal, e. g. if

$$a^{\alpha_1} b_1^{\beta_{11}} b_2^{\beta_{12}} = a^{\alpha_2} b_1^{\beta_{21}} b_2^{\beta_{22}},$$

it follows that

$$a^{\alpha_1 - \alpha_2} b_1^{\beta_{11} - \beta_{21}} b_2^{\beta_{12} - \beta_{22}} = 1.$$

Unless $\alpha_1 = \alpha_2$, $\beta_{11} = \beta_{21}$, $\beta_{12} = \beta_{22}$, this is a new relation (not a product of those

already given) which the generating elements satisfy. But we have seen that no such relation exists. Hence the group generated is actually of order p^3.*

21. It remains to determine for what sets of values of m_1, m_2, n, the groups are identical. To this end, we select in the group G for which m_1, m_2, n have a particular set of values, all the different systems of elements which will generate the group G. For each such system the values of m_1, m_2, n will, as a rule, be different, and the totality of these sets of values is the totality of the values of m_1, m_2, n leading to the group G. We shall thus distinguish all the different admissible sets of values of m_1, m_2, n into classes such that every set of a class, when used as exponents in the generating relations, leads to the same group, and that no two sets belonging to different classes lead to the same group. There will then be just as many different groups as there are classes of values of m_1, m_2, n. We shall find that the classes can be completely distinguished by certain functions of m_1, m_2, n which are invariant for each class, but may vary from class to class.

22. In the group G (defined in Nos. 18, 19) we now make the most general choice of a system of generating elements, a', b'_1, b'_2, viz.

$$a' = a^{\delta} b_1^{\alpha_1} b_2^{\alpha_2}$$
$$b'_i = a^{\gamma_i} b_1^{\beta_{i1}} b_2^{\beta_{i2}} \qquad i = 1, 2.$$

As has been seen (Nos. 18–20), in order that these elements may be a system of generating elements of G they must satisfy relations of the form given in No. 19, and no others not products of these.† That is, we must have

$$a'^p = 1$$
$$b_i'^p = a'^m, \quad i = 1, 2$$
$$b'_2 b'_1 = a'^{n'} b'_1 b'_2$$
$$a' b'_i = b'_i a', \quad i = 1, 2.$$

---

* In the same way it appears in every type subsequently to be considered, and in general, that the given relations with their products and powers are alone sufficient to reduce the group generated to the proper order; that therefore relations that are not products of the given relations reduce the order of the group further, and are hence inadmissible. If (as sometimes happens) relations not products of those given are consequences of them for particular values of the exponents, these values of the exponents, as not leading to a group of the type in question, must be excluded from consideration.

† The group *may* be generated by systems of elements satisfying relations quite different from these, but such systems need not be considered. For we have seen that every group of the type can be generated by a system of elements which does satisfy relations of the form given, and hence if we confine our attention to systems of the latter sort, we shall not thereby have excluded any groups from consideration.

If now the values of $a'b'_i$ in terms of a, b_1, b_2 be substituted, it appears that in order that the relation $a'b'_i = b'_i a'$ may hold it is necessary and sufficient that $\alpha_1 \equiv \alpha_2 \equiv 0 \pmod{p}$.

We may write then

$$a' = a^{\delta}$$
$$b'_i = a^{\gamma} b_1^{\beta_{i1}} b_2^{\beta_{i2}}, \quad i = 1,\ 2.$$

23. In order that the elements a', b'_i may generate the group G_1 it is necessary and sufficient that the elements a, b_i which, by hypothesis, generate G_1 be expressible in terms of a', b'_i. *I. e.*, the following relations must exist:

$$a = a'^{\delta'}$$
$$b_i = a'^{\gamma'_i} b_1'^{\beta'_{i1}} b_2'^{\beta'_{i2}}, \quad i = 1,\ 2.$$

Whence, substituting the values of a', b'_i,

$$a = a^{\delta\delta'}$$
$$b_i = a^{\delta\gamma'_i + f_i(\gamma_1,\, \gamma_2,\, \beta\text{'s})} \cdot b_1^{\beta'_{i1}\beta_{11} + \beta'_{i2}\beta_{21}} \cdot b_2^{\beta'_{i1}\beta_{12} + \beta'_{i2}\beta_{22}},$$

f_i being a polynomial whose precise form is not material here.

From these:

$$\delta\delta' \equiv 1 \pmod{p}$$
$$\delta\gamma'_i + f_i(\gamma_1,\, \gamma_2,\, \beta\text{'s}) \equiv 0 \pmod{p},\ i = 1,\ 2.$$
$$\beta'_{11}\beta_{11} + \beta'_{12}\beta_{21} \equiv 1 \ (\text{“})$$
$$\beta'_{11}\beta_{12} + \beta'_{12}\beta_{22} \equiv 0 \ (\text{“})$$
$$\beta'_{21}\beta_{11} + \beta'_{22}\beta_{21} \equiv 0 \ (\text{“})$$
$$\beta'_{21}\beta_{12} + \beta'_{22}\beta_{22} \equiv 1 \ (\text{“})$$

The integers δ, γ_i, β_{ij} must be so chosen that values of the integers δ', γ'_i, β'_{ij} can be found which will satisfy this system of congruences. To this end, it is necessary and sufficient that the following conditions be satisfied:

$$\delta \not\equiv 0 \pmod{p}$$
$$\begin{vmatrix} \beta_{11}\beta_{12} \\ \beta_{21}\beta_{22} \end{vmatrix} \not\equiv 0 \ (\text{“})$$

No restriction upon γ_1, γ_2 follows, since, whatever their values may be, values of γ'_i can always be found to satisfy the congruences in which γ_1, γ_2 appear.

24. *Notation.**—The symbol $|\alpha|$ shall be used permanently to denote the determinant of the numbers α_{ij}, $i, j = 1, 2 \dots n$.

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*Cf. Smith, Phil. Trans. 1861, p. 293.

If now we choose as generating elements

$$a' = a^{\delta}$$
$$b'_i = a^{\gamma} b_1^{\beta_{i1}} b_2^{\beta_{i2}}, \quad i = 1, 2$$
$$\delta \not\equiv 0, \; |\beta| \not\equiv 0 \pmod{p},$$

the group G will be generated. Substituting these values in the set of relations given in No. 22, the latter reduce to the following, (provided $p > 2$, which we assume hereafter to be the case, reserving the case $p = 2$ for separate consideration):

$$a^{m_1\beta_{i1} + m_2\beta_{i2}} = a^{\delta m'} \quad i = 1, 2.$$
$$a^{n(\beta_{11}\beta_{22} - \beta_{12}\beta_{21})} b_1^{\beta_{11} + \beta_{21}} b_2^{\beta_{12} + \beta_{22}} = a^{\delta n'} b_1^{\beta_{11} + \beta_{21}} . b_2^{\beta_{12} + \beta_{22}}.$$

Whence, if $\bar{\delta}$ be defined by $\bar{\delta}\delta \equiv 1 \pmod{p}$,

$$m'_i \equiv \bar{\delta}(m_1\beta_{i1} + m_2\beta_{i2}) \pmod{p}, \; i = 1, 2.$$
$$n' \equiv \bar{\delta}\,|\beta|\,n \qquad (\;\text{"}\;)$$

By these formulæ are determined all the sets of values of m'_1, m'_2, n' which, when used as exponents in the generating relations, lead to the same group as m_1, m_2, n.

25. The choice of other generating elements for the group may be called a transformation of the group into itself, and the formulæ just written are the corresponding formulæ of transformation for the exponents. Those different sets of values of m_1, m_2, n, and those only, which can be transformed into each other by these formulæ characterize the same group. One of these sets of values may be selected as typical of them all and called the "reduced" set of values. The distinct groups are determined when, by these formulæ, several sets of reduced values are found such that no reduced set can be transformed into any other reduced set, but such that every admissible set of values of m_1, m_2, n can be transformed into one and only one of the reduced sets of values.

26. *Definition.*—An *invariant* of a type is a function of the variable exponents occurring in the generating relations of the type, which remains unaltered in value under any admissible transformation. Two groups for which an invariant has different values cannot be identical because they cannot be transformed into each other by the method just explained.

27. Since neither δ, $|\beta|$, nor n is $\equiv 0$, (mod p), it is always possible to choose δ and $|\beta|$ so that $n' \equiv 1$ (mod p). This is done, for instance, by putting

$$\delta \equiv n, \ |\beta| \equiv \begin{vmatrix} 10 \\ 01 \end{vmatrix} \ (\text{mod } p)^*$$

when

$$m_i' \equiv \bar{n} m_i, \quad i = 1, 2, \quad n' \equiv 1 \ (\text{mod } p).$$

This means that if a group be characterized by exponents m_1, m_2, n, it can always, by a different choice of generating elements, be expressed in a form having the exponents $\bar{n}m_1$, $\bar{n}m_2$, 1, i. e. by

$$\begin{aligned} a^p &= 1 \\ b_i^p &= a^{\bar{n}m}, \quad i = 1, 2, \\ b_2 b_1 &= a b_1 b_2. \end{aligned}$$

Since all groups of the type are contained in this form, we confine our attention hereafter to groups in which $n \equiv 1$ (mod p) and apply only such transformations as will leave n unaltered. This is no restriction upon the generality of the treatment.

Since by the results of No. 24, m_i' can be expressed linearly and homogeneously in terms of m_i (mod p), and conversely, it follows that the greatest common divisor of m_1, m_2 and p remains unaltered under any transformation, i. e. G.C.D. $m_1, m_2, p =$ G.C.D. m_1', m_2', p.

28. *Notation.*†—Denote the greatest common divisor of $a, b, \ldots . l$, by $[a, b, \ldots . l]$.

$A = [m_1, m_2, p]$ is then an *invariant* of the type. That two groups be identical it is *necessary* that A have the same value for each. We proceed to show that it is also *sufficient.*

A may have the values 1 and p. If $A = 1$, i. e. m_1 and m_2 are not both $\equiv 0$ (mod p), δ and β_{ij} can always be so chosen that

$$\begin{aligned} m_1'' &\equiv \bar{\delta}(m_1'\beta_{11} + m_2'\beta_{12}) \equiv \bar{n}\bar{\delta}(m_1\beta_{11} + m_2\beta_{12}) \equiv 1 \ (\text{mod } p) \\ m_2'' &\equiv \bar{\delta}(m_1'\beta_{21} + m_2'\beta_{22}) \equiv \bar{n}\bar{\delta}(m_1\beta_{21} + m_2\beta_{22}) \equiv 0 \ (\quad \text{“} \quad) \\ n'' &\equiv \bar{\delta}|\beta|n' \equiv \bar{\delta}|\beta| \equiv 1 \qquad\qquad\qquad\qquad\qquad (\quad \text{“} \quad). \end{aligned}$$

* This notation means that the elements of $|\beta|$ are respectively congruent (mod p) to the values written.

† Smith, Phil. Trans. 1861, p. 321.

I. e., if $A=1$, all the sets of values of m_i n admissible under this restriction can be reduced to $m_1\equiv 1$, $m_2\equiv 0$, $n\equiv 1 \pmod p$. In other words, all the different sets of values of m_i, n for which $A=1$, lead to the same group. When $A=p$, there is only one set of values of m_i admissible, viz. $m_1\equiv m_2\equiv 0 \pmod p$. There are then only two distinct groups according as $A=1$ or $A=p$.

We take as reduced values

$$A=1,\ m_1\equiv 1,\ m_2\equiv 0,\ n\equiv 1 \pmod p$$
$$A=p,\ m_1\equiv 0,\ m_2\equiv 0,\ n\equiv 1\ (\ \text{“}\ ).$$

The groups themselves are generated by elements satisfying the relations:

$$\begin{array}{lll} (A=1) & & (A=p) \\ a^p=1 & & a^p=1 \\ b_1^p=a & & b_1^p=1 \\ b_2^p=1 & & b_2^p=1 \\ b_2b_1=ab_1b_2 & & b_2b_1=ab_1b_2 \\ ab_i=b_ia \quad i=1,\,2 & & ab_i=b_ia \quad i=1,\,2. \end{array}$$

29. *Case* $p=2$.

When $p=2$, instead of the results of No. 24, the following relations are deduced in a similar manner; (remembering that since $\delta \not\equiv 0 \pmod{p=2}$ $|\beta|\not\equiv 0$, $n\not\equiv 0 \pmod 2$ we have $\delta\equiv\bar\delta\equiv 1$, $n\equiv 1$, $|\beta|\equiv 1 \pmod 2$):

$$m_i'\equiv\beta_{i1}\beta_{i2}+\beta_{i1}m_1+\beta_{i2}m_2 \pmod 2,$$
$$n'\equiv n\equiv 1 \qquad\qquad (\ \text{“}\ ).$$

If $\qquad m_1\equiv m_2\equiv 1 \pmod 2$ then

$$m_1'\equiv\beta_{11}\beta_{12}+\beta_{11}+\beta_{12} \pmod 2$$
$$m_2'\equiv\beta_{21}\beta_{22}+\beta_{21}+\beta_{22} \pmod 2.$$

Since $|\beta|\equiv 1 \pmod 2$, β_{i1} and β_{i2} are not both $\equiv 0 \pmod 2$, and hence necessarily

$$m_1'\equiv m_2'\equiv 1 \pmod 2.$$

Since the transformation is invertible m_1m_2 remains unaltered (mod 2) under any admissible transformation. I. e., $A\equiv m_1m_2 \pmod 2$ is an invariant. We have considered above the case $A\equiv 1 \pmod 2$. If $A\equiv 0 \pmod 2$ at least one m is $\equiv 0 \pmod 2$, and the β's can always be so chosen that $m_1'\equiv m_2'\equiv 0 \pmod 2$.

There are then two groups according as $A\equiv 0$ or $A\equiv 1 \pmod 2$.

30. *Groups of order p^4.*

$$\textit{Types}:\quad \begin{array}{rl} \text{I.} & \left.\begin{array}{r}(1111)\\(211)\\(22)\\(31)\\(4)\end{array}\right\}\text{ The commutative groups.}\\ \text{II.} & (1\between111)\\ \text{III.} & (2\between11)\\ \text{IV.} & (1\between21)\\ \text{V.} & (11\between11)\\ \text{VI.} & (1\between1\between11). \end{array}$$

31. *Type* $(1\between111)$.

By the methods already used, (Nos. 18, 19), this type is seen to be generated by elements satisfying the relations:

$$\begin{aligned} &a^p = 1\\ &b_i^p = a^{m_i},\ i = 1, 2, 3\\ &b_j b_i = a^{n_{ij}} b_i b_j,\ i, j = 1, 2, 3 \text{ independently.}\\ &a b_i = b_i a,\ i = 1, 2, 3. \end{aligned}$$

From the third relation it follows that

$$n_{ij} \equiv -n_{ji},\quad n_{ii} \equiv 0 \pmod p.$$

No new relations are found by transforming the above relations.

Formulæ of multiplication:

$$b_j^{\beta_2} b_i^{\beta_1} = a^{\beta_1\beta_2 n_{ij}} b_i^{\beta_1} b_j^{\beta_2}$$

$$(a^\alpha b_1^{\beta_1} b_2^{\beta_2} b_3^{\beta_3})^q = a^{q\alpha + \frac{q(q-1)}{2} \sum\limits_{j=i+1}^{3} \sum\limits_{i=1,2} \beta_i\beta_j n_{ij}} . \, b_1^{q\beta_1} b_2^{q\beta_2} b_3^{q\beta_3}.$$

32. *Condition* that no element except a and its powers be commutative with every element of the group. That $c = a^\alpha b_1^{\gamma_1} b_2^{\gamma_2} b_3^{\gamma_3}$ be commutative with every element of the group, it is necessary and sufficient that

$$c b_i = b_i c,\quad i = 1, 2, 3.$$

Substituting the value of c in this relation and reducing by the formulæ for multiplication, we find that it reduces to

$$a^{\sum_{j=i+1}^{3} \gamma_j n_{ij}} = a^{\sum_{j=1}^{i-1} \gamma_j n_{ji}} \quad i=1,\ 2,\ 3.$$

Whence

$$\sum_{j=i+1}^{3} \gamma_j n_{ij} \equiv \sum_{j=1}^{i-1} \gamma_j n_{ji} \pmod p \quad i=1,\ 2,\ 3,$$

or, remembering that $n_{ji} \equiv -n_{ij}$, $n_{ii} \equiv 0 \pmod p$,

$$\sum_{j=1}^{3} \gamma_j n_{ij} \equiv 0 \pmod p \quad i=1,\ 2,\ 3.$$

In order that none but a and its powers may be commutative with every element of the whole group, the system of congruences last written must have no other solutions than

$$\gamma_1 \equiv \gamma_2 \equiv \gamma_3 \equiv 0 \pmod p.$$

The n's must therefore be so chosen that the system of congruences can be satisfied only by these values of γ_i. The condition for this is:

$$|n| \not\equiv 0 \pmod p.$$

But $|n|$ is a skew-symmetric determinant of odd order and hence*

$$|n| \equiv 0 \pmod p.$$

The type is therefore non-existent.

33. *Type* (2, 1, 1).

By the previous methods (Nos. 18, 19), it is found that this type is generated by elements a, b_1, b_2, satisfying the relations:

$$\begin{aligned} a^{p^2} &= 1 \\ b_1^p &= a^{m_1} \\ b_2^p &= a^{m_2} \\ b_2 b_1 &= a^n b_1 b_2 \\ a b_i &= b_i a, \quad i=1,\ 2. \end{aligned}$$

* Baltzer, Determinanten, §5, 8.

Formulæ of multiplication:

$$b_2^{\beta_2} b_1^{\beta_1} = a^{\beta_1 \beta_2 n} b_1^{\beta_1} b_2^{\beta_2}$$

$$(a^{\alpha} b_1^{\beta_1} b_2^{\beta_2})^q = a^{q\alpha + \frac{q(q-1)}{2}\beta_1\beta_2 n} . b_1^{q\beta_1} . b_2^{q\beta_2} .$$

Relations consequent on these by transformation. Transforming the second relation written by b_2, and reducing by the multiplication formulæ, we have

$$a^{np} b_1^p = a^{m_1} ;$$

whence $$a^{np} = 1 .$$

Accordingly, that this may not be a new relation satisfied by a, n must be so chosen that (v. note to No. 20)

$$np \equiv 0 \pmod{p^2}$$
$$\therefore \quad n \equiv 0 \pmod{p} .$$

The condition that no element other than a and its powers be commutative with every element of the group is seen, by the method of No. 32, to be

$$n \not\equiv 0 \pmod{p^2} .$$

Nothing further is found by transforming the remaining relations.

34. If, now, the following elements of the group be considered as generating elements

$$a' = a^{\delta}$$
$$b_i' = a^{\alpha_i} b_1^{\beta_{i1}} b_2^{\beta_{i2}}, \quad i = 1, 2,$$

it is proved, by the method already explained in detail (Nos. 21 to 23), that the necessary and sufficient condition that the group generated by a', b_i' be actually the group under consideration, and not a subgroup of it, are

$$\delta \not\equiv 0, \ |\beta| \not\equiv 0 \pmod{p}, \ \alpha_i \text{ remaining entirely arbitrary.}$$

By the method explained in No. 24 are found the following

Formulæ of Transformation:

$$\bar{\delta}\left(p\alpha_i + \frac{p(p-1)}{2}\beta_{i1}\beta_{i2}n + \beta_{i1}m_1 + \beta_{i2}m_2\right) \equiv m_i' \pmod{p^2}, \ i = 1, 2,$$

$$\bar{\delta}\, |\beta|\, n \equiv n' \pmod{p^2} .$$

Since both n and n' are necessarily divisible by p but not by p^2, we can always choose δ and $|\beta|$ so that $n' \equiv p \pmod{p^2}$, (e. g. by

$$\delta \equiv \frac{n}{p}, \ |\beta| \equiv \begin{vmatrix} 10 \\ 01 \end{vmatrix} \pmod{p}).$$

It appears from the form of the first formula of transformation above that the G.C.D. of m_1, m_2, p is an invariant, and that by a proper choice of the arbitrary quantity α_i, m'_i can always be made less than p. Since n is divisible by p, the case $p = 2$ is included in the general case.

We distinguish then

$$A = [m_1, m_2, p] = 1, \text{ and } A = p.$$

If $A = 1$, β_{ij} can always, as in the type $(1)(11)$ (No. 28), be so chosen that $m_1 \equiv 1$, $m_2 \equiv 0 \pmod{p}$, and by giving the arbitrary quantity α_i a suitable value, we can always make $m'_1 \equiv 1$, $m'_2 \equiv 0 \pmod{p^2}$; while if $A = p$, i. e. if $m_1 \equiv m_2 \equiv 0 \pmod{p}$, we can always, by a proper choice of α_i, make $m'_1 \equiv m'_2 \equiv 0 \pmod{p^2}$. There are therefore two groups of this type, according as

$$A = 1, \text{ or } A = p.$$

35. *Type* $(1)(21)$.

This type is generated by elements a, b_1, b_2, satisfying the relations:

$$\begin{aligned} a^p &= 1 \\ b_1^{p^2} &= a^{m_1} \\ b_2^p &= a^{m_2} \\ b_2 b_1 &= a^n b_1 b_2 \\ a b_i &= b_i a, \ i = 1, 2. \end{aligned}$$

Formulæ of multiplication:

$$b_2^{\beta_2} b_1^{\beta_1} = a^{\beta_1\beta_2 n} b_1^{\beta_1} b_2^{\beta_2}$$

$$(a^a b_1^{\beta_1} b_2^{\beta_2})^q = a^{qa + \frac{q(q-1)}{2}\beta_1\beta_2 n} . b_1^{q\beta_1} b_2^{q\beta_2}.$$

By the formulæ of multiplication,

$$b_2 b_1^p = a^{pn} b_1^p b_2$$

$$\text{or} \quad b_2 b_1^p = b_1^p b_2.$$

That is, b_1^p is commutative with b_2, and being already commutative with itself and with a, it is commutative with every element of the group. This is contrary to the hypothesis that a and its powers constitute the totality of the elements commutative with every element of the group. There are therefore no groups of this type.

36. *Type* $(11\chi 11)$.

By the previous methods, this type is seen to be generated by elements a_1, a_2, b_1, b_2, satisfying the relations

$$\left.\begin{aligned} a_i^p &= 1 \\ b_i^p &= a_1^{m_{i1}} a_2^{m_{i2}} \end{aligned}\right\} i = 1,\ 2.$$

$$b_2 b_1 = a_1^{n_1} a_2^{n_2} b_1 b_2$$

$$a_1 a_2 = a_2 a_1$$

$$a_i b_j = b_j a_i, \quad i, j = 1,\ 2 \text{ independently.}$$

Formulæ of multiplication:

$$b_2^{\beta_2} b_1^{\beta_1} = a_1^{\beta_1\beta_2 n_1} . a_2^{\beta_1\beta_2 n_2} . b_1^{\beta_1} . b_2^{\beta_2}$$

$$(a_1^{\alpha_1} a_2^{\alpha_2} b_1^{\beta_1} b_2^{\beta_2})^q = a_1^{q\alpha_1 + \frac{q(q-1)}{2}\beta_1\beta_2 n_1} . a_2^{q\alpha_2 + \frac{q(q-1)}{2}\beta_1\beta_2 n_2} . b_1^{q\beta_1} b_2^{q\beta_2}.$$

No new conditions are found by transforming the given relations by each of the generating elements.

In order that no element, other than powers and products of a_1, a_2, be commutative with every element of the group, it is shown as before (No. 32), to be necessary and sufficient that at least one of the quantities n_1, n_2 be not congruent zero, (mod p).

37. Just as before (Nos. 21–24), the conditions and formulæ of transformation are found. It is necessary to distinguish the cases $p > 2$ and $p = 2$.

Formulæ of transformation, $p > 2$.

$$\sum_{j=1,2} \beta_{ij} m_{j\kappa} \equiv \sum_{j=1,2} \delta_{j\kappa} m'_{ij} \pmod p \quad i, \kappa = 1,\ 2 \text{ independently.}$$

$$(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}) n_i \equiv \delta_{1i} n'_1 + \delta_{2i} n'_2 \pmod p \quad i = 1,\ 2.$$

$$|\beta| \not\equiv 0, \quad |\delta| \not\equiv 0, \pmod p.$$

From the second set of formulæ,

$$n_1' \equiv \overline{|\delta|} \cdot |\beta| (n_1\delta_{22} - n_2\delta_{21}), \pmod p$$
$$n_2' \equiv \overline{|\delta|} \cdot |\beta| (n_2\delta_{11} - n_1\delta_{12}), \;(\quad \text{“} \quad).$$

We choose the β's and δ's so that

$$n_1' \equiv 1, \; n_2' \equiv 0, \pmod p.$$

Such a choice, for instance, is

$$|\beta| \equiv \begin{vmatrix} 10 \\ 01 \end{vmatrix}, \pmod p$$
$$|\delta| \equiv \begin{vmatrix} n_1 & n_2 \\ \delta_{21} & \delta_{22} \end{vmatrix} \equiv 1, \pmod p.$$

With these values, the resulting values of m'_{ij} are

$$m'_{11} \equiv \begin{vmatrix} m_{11} & \delta_{21} \\ m_{12} & \delta_{22} \end{vmatrix}, \pmod p,$$
$$m'_{12} \equiv \begin{vmatrix} n_1 & m_{11} \\ n_2 & m_{12} \end{vmatrix}, \;(\quad \text{“} \quad),$$
$$m'_{21} \equiv \begin{vmatrix} m_{21} & \delta_{21} \\ m_{22} & \delta_{22} \end{vmatrix}, \;(\quad \text{“} \quad),$$
$$m'_{22} \equiv \begin{vmatrix} n_1 & m_{21} \\ n_2 & m_{22} \end{vmatrix}, \;(\quad \text{“} \quad).$$

In applying any further transformation we must have, in order to leave the n's unaltered,

$$|\beta| \equiv \delta_{11}, \; \delta_{12} \equiv 0, \pmod p.$$

$A = [|m|, p]$ is invariant. This is proved by applying to the m's the most general transformation admissible. The determinant of the m''s is found to differ from that of the m's by a factor which is not congruent zero, (mod p). Hence A remains unaltered.

38. Suppose $A = 1$, i. e. $|m| \not\equiv 0 \pmod p$. Applying to the m''s the most general transformation leaving the n''s unaltered, the result is (see conditions above):

$$m''_{11} \equiv \overline{|\beta|} \{\beta_{11}m'_{11} + \beta_{12}m'_{21} - \delta_{21}\bar{\delta}_{22}(\beta_{11}m'_{12} + \beta_{12}m'_{22})\} \pmod p,$$
$$m''_{21} \equiv \overline{|\beta|} \{\beta_{21}m'_{11} + \beta_{22}m'_{21} - \delta_{21}\bar{\delta}_{22}(\beta_{21}m'_{12} + \beta_{22}m'_{22})\} \;(\quad \text{“} \quad),$$
$$m''_{12} \equiv \bar{\delta}_{22}(\beta_{11}m'_{12} + \beta_{12}m'_{22}) \;(\quad \text{“} \quad),$$
$$m''_{22} \equiv \bar{\delta}_{22}(\beta_{21}m'_{12} + \beta_{22}m'_{22}) \;(\quad \text{“} \quad).$$

Choose $|\beta| \equiv \begin{vmatrix} m'_{22}, & -m'_{12} \\ -m'_{21}, & m'_{11} \end{vmatrix}$, $\delta_{21} \equiv 0$, $\delta_{22} \equiv |m|$, (mod p).

then $\quad m''_{11} \equiv 1$, $m''_{12} \equiv 0$, (mod p),

$\qquad m''_{21} \equiv 0$, $m''_{22} \equiv 1$, (").

I. e., all values whatever of the m's, subject to the condition that $|m| \not\equiv 0$ (mod p) can be reduced to the set just written, by a transformation leaving $n_1 \equiv 1$, $n_2 \equiv 0$. There is hence but one group when $A = 1$.

39. Suppose $A = p$, i. e., $|m| \equiv 0$ (mod p). In deducing the first four congruences of No. 38, no assumption was made as to whether or not $|m| \equiv 0$ (mod p). These formulæ, therefore, hold true also in the present case. On the face of the formulæ it is evident that $B = [m'_{12}, m'_{22}, p]$ remains unaltered by this transformation. When the values of m'_{12} and m'_{22} in terms of the original parameters are substituted, we must have an invariant of the type. For if

$$B = \left[\begin{vmatrix} n_1, & m_{11} \\ n_2, & m_{12} \end{vmatrix}, \begin{vmatrix} n_1, & m_{21} \\ n_2, & m_{22} \end{vmatrix}, p \right]$$

is not invariant, it is possible by some admissible transformation applied to n_1, n_2, m_{ij}, to reach values $n_1^{(1)}$, $n_2^{(1)}$, $m_{ij}^{(1)}$, such that $B^{(1)} \neq B$. When now $n_1^{(1)}$ is made unity and $n_2^{(1)}$ zero (mod p), $m_{12}^{(1)\prime}$ and $m_{22}^{(1)\prime}$ are the determinants in $B^{(1)}$. By the hypothesis that $B^{(1)} \neq B$, they have not the same G.C.D. with p as the determinants in B.

Therefore, supposing B *not* an invariant of the type, it is possible by three transformations, viz. from m'_{ij} back to m_{ij}, then to $m_{ij}^{(1)}$, then to $m_{ij}^{(1)\prime}$, (and these transformations can be combined into one leaving $n'_1 \equiv 1$, $n'_2 \equiv 0$, (mod p)) to alter $B = [m'_{12}, m'_{22}, p]$. But this last has been shown to be impossible. Hence B is an invariant of the type.

40. The same reasoning shows generally that if, after having applied any number of particular transformations, any function of the resulting parameters remains unaltered under every transformation which does not affect the results of the previous transformations, then, when the transformed parameters are replaced by their values in terms of the original parameters, the resulting expression is an invariant of the type.

41. That B is an invariant can also be verified by applying to it the most general admissible transformation, when it appears that B remains unaltered.

We distinguish now $B=1$, and $B=p$. Case $B=1$. As a particular transformation we choose,

$$|\beta| \equiv \begin{vmatrix} m'_{22}, & -m'_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix} \equiv 1, \quad \begin{array}{l} \delta_{22} \equiv 1 \pmod{p}, \\ \delta_{21} \equiv \beta_{21} m'_{11} + \beta_{22} m'_{21}, \pmod{p}, \end{array}$$

while we have in addition (No. 37) the conditions $\delta_{11} \equiv |\beta| \equiv 1$, $\delta_{12} \equiv 0 \pmod{p}$.

These values, substituted in the congruences of No. 38 give, (remembering that $|m| \equiv 0 \pmod{p}$),

$$\begin{array}{ll} m''_{11} \equiv \overline{|\beta|}(m'_{22}m'_{11} - m'_{12}m'_{21}) \equiv 0 & \pmod{p} \\ m''_{21} \equiv 0, \; m''_{12} \equiv 0, \; m''_{22} \equiv 1 & (\quad \text{"} \quad). \end{array}$$

There is then only one group in case $A=p$, $B=1$.

If $B=p$, i. e. $m'_{12} \equiv m'_{22} \equiv 0$, (mod p) we must distinguish whether or not $m'_{11} \equiv m'_{21} \equiv 0 \pmod{p}$. This leads to a new invariant, viz.

$$C = [m_{11}, m_{12}, m_{21}, m_{22}, p].$$

That this is invariant appears from the first formula of No. 37, remembering that $|\delta| \equiv\!\!\!/ \; 0 \pmod{p}$.

If $B=p$, $C=1$, we make $m''_{11} \equiv 0$, $m''_{21} \equiv 1 \pmod{p}$, (No. 38).

If $B=p$, $C=p$, then necessarily

$$m''_{11} \equiv m''_{21} \equiv 0 \pmod{p}.$$

We have now found three invariants of the type. We have proved that all the groups characterized by any one of the possible combinations of values of the invariants are identical. Two groups whose invariants do not coincide in value cannot be identical. We have then reached a complete system of invariants which defines every group of the type once and only once when the invariants take all possible values.

Groups, Type $(11)(11)$, $p > 2$.

	1.)	2.)	3.)	4.)
$A=$	1,	p,	p,	p,
$B=$	1,	1,	p,	p,
$C=$	1,	1,	1,	p.

42. *Type* (11)(11), $p=2$.

In this case we find as before the *formulæ of transformation:*

$$\beta_{i1}\beta_{i2}n_\kappa + \beta_{i1}m_{1\kappa} + \beta_{i2}m_{2\kappa} \equiv m'_{i1}\delta_{1\kappa} + m'_{i2}\delta_{2\kappa} \pmod 2, \quad i,\ \kappa = 1,\ 2, \text{ independently.}$$

$$(\beta_{11}\beta_{22} - \beta_{12}\beta_{21})n_\kappa \equiv \delta_{1\kappa}n'_1 + \delta_{2\kappa}n'_2 \pmod 2, \quad \kappa = 1,\ 2.$$

$$|\delta| \not\equiv 0, \quad |\beta| \not\equiv 0 \pmod 2 \;\therefore\; |\delta| \equiv |\beta| \equiv 1 \pmod 2.$$

At least one of the integers $n_1 n_2$ is $\not\equiv 0 \pmod 2$. From the second formula of transformation,

$$n'_1 \equiv \begin{vmatrix} n_1 & \delta_{21} \\ n_2 & \delta_{22} \end{vmatrix} \pmod 2,$$

$$n'_2 \equiv \begin{vmatrix} \delta_{11} & n_1 \\ \delta_{12} & n_2 \end{vmatrix} \quad (\text{ `` })$$

$$\text{Choose } \delta \equiv \begin{vmatrix} n_1, & n_2 \\ n_1 + n_2, & n_1 \end{vmatrix}, \quad |\beta| \equiv \begin{vmatrix} 10 \\ 01 \end{vmatrix} \pmod 2.$$

This makes $n'_1 \equiv 1$, $n'_2 \equiv 0$, (mod 2), irrespective of the values of n_1 and n_2. The resulting values of m'_{ij} are

$$m'_{11} \equiv \begin{vmatrix} m_{11}, & n_1 + n_2 \\ m_{12}, & n_1 \end{vmatrix} \pmod 2,$$

$$m'_{12} \equiv \begin{vmatrix} n_1, & m_{11} \\ n_2, & m_{12} \end{vmatrix} \quad (\text{ `` }),$$

$$m'_{21} \equiv \begin{vmatrix} m_{21}, & n_1 + n_2 \\ m_{22}, & n_1 \end{vmatrix} \quad (\text{ `` }),$$

$$m'_{22} \equiv \begin{vmatrix} n_1, & m_{21} \\ n_2, & m_{22} \end{vmatrix} \quad (\text{ `` }).$$

The most general transformation leaving the values $n_1 \equiv 1$, $n_2 \equiv 0 \pmod 2$ unaltered is

$$|\delta| \equiv \begin{vmatrix} 1 & 0 \\ \delta_{21} & 1 \end{vmatrix}, \quad |\beta| \text{ arbitrary.}$$

Applying this transformation we find that

$$m''_{11} \equiv \beta_{11}\beta_{12} + \beta_{11}m'_{11} + \beta_{12}m'_{21} + \delta_{21}(\beta_{11}m'_{12} + \beta_{12}m'_{22}) \pmod 2,$$
$$m''_{21} \equiv \beta_{21}\beta_{22} + \beta_{21}m'_{11} + \beta_{22}m'_{21} + \delta_{21}(\beta_{21}m'_{12} + \beta_{22}m'_{22}) \quad (\text{ `` }),$$
$$m''_{12} \equiv \beta_{11}m'_{12} + \beta_{12}m'_{22} \pmod 2,$$
$$m''_{22} \equiv \beta_{21}m'_{12} + \beta_{22}m'_{22} \quad (\text{ `` }).$$

43. Since m''_{12} and m''_{22} are expressible linearly and homogeneously in terms of m'_{12} and m'_{22} and conversely, it follows that $[m'_{12},\ m'_{22},\ 2]$ remains invariant

under this, the most general transformation leaving $n_1' \equiv 1$, $n_2' \equiv 0$, (mod 2) and that hence, in terms of the original parameters,

$$A \equiv \left[\begin{vmatrix} n_1, & m_{11} \\ n_2, & m_{12} \end{vmatrix}, \quad \begin{vmatrix} n_1, & m_{21} \\ n_2, & m_{22} \end{vmatrix}, \right] \pmod 2$$

is an invariant of the type according to the considerations of Nos. 39, 40. The fact that A is an invariant can also be verified by applying to A the most general transformation admissible, when it is found to remain unaltered. If, when $A \equiv 0$, $m_{11}' \equiv m_{21}' \equiv 1$, (mod 2), then necessarily, since $|\beta| \equiv 1$ (mod 2), $m_{11}'' \equiv m_{21}'' \equiv 1$, (mod 2). In this case, therefore, we must distinguish whether or not $m_{11}' \equiv m_{21}' \equiv 1$, (mod 2), and hence $m_{11}' . m_{21}'(A+1)$ is invariant. Put

$$B \equiv \begin{vmatrix} m_{11}, & n_1 + n_2 \\ m_{12}, & n_1 \end{vmatrix} . \begin{vmatrix} m_{21}, & n_1 + n_2 \\ m_{22}, & n_1 \end{vmatrix} . (A+1), \pmod 2.$$

This is invariant according to the general theory, (Nos. 39, 40), as may also be verified by applying the most general transformation admissible.

We have just considered above the case $A \equiv 0$, $B \equiv 1$, (mod 2), while if $A \equiv 0$, $B \equiv 0$, (mod 2), we can always, by a proper choice of $|\beta|$ make $m_{11}'' \equiv m_{21}'' \equiv 0$ (mod 2). If $A \equiv 0$, there are therefore two groups according as $B \equiv 0, 1$ (mod 2).

Case $A \equiv 1$. In this case $B \equiv 0$ (mod 2) necessarily. Applying the transformation

$$|\beta| \equiv \begin{vmatrix} m_{22}', & m_{12}' \\ m_{22}' + 1, & 1 \end{vmatrix}, \quad |\delta| \equiv \begin{vmatrix} 1, & 0 \\ \delta_{21}, & 1 \end{vmatrix} \pmod 2,$$

with the result:

$$m_{12}'' \equiv 0, \ m_{22}'' \equiv 1 \pmod 2,$$
$$m_{11}'' \equiv m_{12}'m_{22}' + m_{11}'m_{22}' + m_{12}'m_{21}' \pmod 2,$$
$$m_{21}'' \equiv (m_{22}' + 1)(m_{11}' + 1) + m_{21}' + \delta_{21} \pmod 2;$$

m_{21}'' can always, by a proper choice of the arbitrary quantity δ_{21}, be made $\equiv 0$ (mod 2). We must distinguish, finally, whether or not $m_{11}'' \equiv 0$ (mod 2). By previous methods, then,

$$C \equiv m_{11}'' \equiv m_{12}'m_{22}' + |m'|$$
$$\equiv \begin{vmatrix} n_1 & m_{11} \\ n_2 & m_{12} \end{vmatrix} . \begin{vmatrix} n_1 & m_{21} \\ n_2 & m_{22} \end{vmatrix} + \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \pmod 2$$

is an invariant of the type, and distinguishes the two groups possible when $A \equiv 1$ (mod 2).

We have now considered all the possibilities, have reached a complete system of invariants and can enumerate thereby the groups of the type, viz.

Groups, Type (11)(11), $p = 2$.

	1.)	2.)	3.)	4.)
$A =$	0,	1,	1,	0,
$B =$	1,	0,	0,	0,
$C =$	0,	1,	0,	0.

44. *Type* (1)(1)(11).

The groups of this type are generated by four elements, a, b, c_1, c_2, satisfying the relations:

$$\begin{aligned} a^p &= 1, \\ b^p &= a^m, \\ \left.\begin{aligned} c_i^p &= a^{n_i} b^{f_i}, \\ c_i b &= a^{q} b c_i, \end{aligned}\right\} & \; i = 1,\ 2. \\ c_2 c_1 &= a^t b^h c_1 c_2, \\ ab &= ba, \\ ac_i &= c_i a, \quad i = 1,\ 2. \end{aligned}$$

It is necessary that $h \not\equiv 0 \pmod{p}$. For otherwise the group $G^{(1)}$ whose relations are found (No. 12) by placing in the above relations $a = 1$ would be a commutative group, and not of type (1)(11) as it should be.

Formulæ of multiplication:

$$c_i^\sigma b^\rho = a^{\sigma\rho q_i} b^\rho c_i^\sigma,$$

$$c_2^\sigma c_1^\rho = a^{\sigma\rho\left[t + h\frac{(\sigma-1)q_2 + (\rho-1)q_1}{2}\right]} . b^{\rho\sigma h} c_1^\rho c_2^\sigma,$$

$$(b^\beta c_1^{\gamma_1} c_2^{\gamma_2})^\tau = a^{\frac{\tau(\tau-1)}{2}\left\{\gamma_1\gamma_2\left[t + h\frac{(\gamma_1-1)q_1 - q_2}{2}\right] + (\gamma_1 q_1 + \gamma_2 q_2)\beta_1\right\} + \frac{\tau(\tau-1)(2\tau-1)}{6} h\gamma_1\gamma_2\left[\gamma_1 q_1 + \frac{\gamma_2 q_2}{2}\right]}$$
$$. b^{\frac{\tau(\tau-1)}{2}\gamma_2\gamma_1 h + \tau\beta} . c_1^{\tau\gamma_1} c_2^{\tau\gamma_2}.$$

It will be seen by the methods used previously that in order that no element except a and its powers be commutative with every element of the whole group, it is necessary and sufficient that not both q_1 and q_2 be congruent zero (mod p).

45. By transforming all the generating relations by each of the generating elements of the group, we find the complete system of relations consequent upon

the given relations. Nearly all of the relations resulting reduce to one or the other of those given, but the relations obtained by transforming the third relation given by c_j do not do so, and when $p > 2$, they are

$$\begin{aligned} a^{-mh+f_2q_1} &= 1, \\ a^{-mh+f_1q_2} &= 1, \\ a^{f_2q_2} &= 1, \\ a^{f_1q_1} &= 1. \end{aligned}$$

Hence, (see note to No. 20),

$$\begin{aligned} mh - f_2q_1 &\equiv 0 \pmod{p}, \\ mh - f_1q_2 &\equiv 0 \;(\;\;\text{"}\;\;), \\ f_2q_2 &\equiv 0 \;(\;\;\text{"}\;\;), \\ f_1q_1 &\equiv 0 \;(\;\;\text{"}\;\;). \end{aligned}$$

From these congruences it follows, since q_1 and q_2 are not both congruent zero, (mod p), that

$$m \equiv f_1 \equiv f_2 \equiv 0 \pmod{p}.$$

The case $p = 2$ is reserved for separate consideration.

Substituting these values, the generating relations become (provided $p > 2$):

$$\begin{aligned} a^p &= 1, \\ b^p &= 1, \\ \left.\begin{aligned} c_i^p &= a^{n_i}, \\ c_ib &= a^{q_i}bc_i, \end{aligned}\right\} & \; i = 1,\ 2. \\ c_2c_1 &= a^tb^hc_1c_2, \\ ab &= ba, \\ ac_i &= c_ia. \quad i = 1,\ 2. \end{aligned}$$

Subject to the restrictions $h \not\equiv 0$, q_1 and q_2 not both $\equiv 0$, (mod p), these relations define a group of our type for every set of values of n_i, q_i, t, h.

46. We now make the most general choice of generating elements a', b', c_i', as follows (cf. No. 22):

$$\begin{aligned} a' &= a^\delta, \\ b' &= a^\epsilon b^\zeta, \\ c_i' &= a^{\alpha_i}b^{\beta_i}c_1^{\gamma_{i1}}c_2^{\gamma_{i2}}, \quad i = 1,\ 2. \end{aligned}$$

In order that the group generated by a', b', c_i', be actually the group under

consideration, and not a subgroup of it, it appears, by previous methods (No. 23), to be necessary and sufficient that the conditions

$$\delta \not\equiv 0, \ \zeta \not\equiv 0, \ |\gamma| \not\equiv 0 \ (\text{mod } p),$$

be satisfied.

At this point, distinction must be made between $p = 3$ and $p > 3$. Reserving the case $p = 3$ for subsequent consideration, it is found by previous methods (Nos. 22–25) that, when $p > 3$, the following are the

Formulæ of transformation:

$$h' \equiv \bar{\zeta}|\gamma|h \ (\text{mod } p),$$

$$\left.\begin{array}{l} n_i' \equiv \bar{\delta}(\gamma_{i1}n_1 + \gamma_{i2}n_2) \ (\text{mod } p), \\ q_i' \equiv \bar{\delta}\zeta(\gamma_{i1}q_1 + \gamma_{i2}q_2) \ (\quad \text{``} \quad), \end{array}\right\} i = 1, \ 2,$$

$$t' \equiv \bar{\delta}\Big[\, |\gamma|t - \varepsilon\bar{\zeta}|\gamma|h + \Big\{h\Big[\frac{\gamma_{11}^2\gamma_{22} - \gamma_{12}\gamma_{21}^2 - |\gamma|}{2} + \gamma_{11}\gamma_{22}(\gamma_{22} - \gamma_{12})\Big] + \gamma_{21}\beta_1 - \gamma_{11}\beta_2\Big\}q$$

$$+ \Big\{h\Big[\frac{\gamma_{11}\gamma_{22}^2 - \gamma_{21}\gamma_{12}^2 - |\gamma|}{2}\Big] + \gamma_{22}\beta_1 - \gamma_{12}\beta_2\Big\}q_2\Big].$$

47. Since ε is perfectly arbitrary, and appears only in the value of t' and is there multiplied by a factor which can never be congruent zero (mod p), it follows that, in every transformation, ε can (without affecting any quantity other than t') be given such a value that $t' \equiv 0$ (mod p). We shall suppose throughout that this is done.

We first make $h' \equiv 1$ (mod p) by the following transformation:

$$|\gamma| \equiv \begin{vmatrix} 10 \\ 01 \end{vmatrix}, \ \varepsilon \equiv \beta_1 \equiv \beta_2 \equiv 0, \ \delta \equiv 1, \ \zeta \equiv h, \ (\text{mod } p).$$

Results:

$$n_i' \equiv n_i, \ q_i' \equiv hq_i, \ t' \equiv t + hq_1, \ h' \equiv 1, \ (\text{mod } p).$$

If now, considering the primed exponents as given, we apply the most general transformation such that $h'' \equiv h' \equiv 1$ (mod p), the exponents of transformation are subject to the restriction $|\gamma| \equiv \zeta$ (mod p). The results of this transformation are

$$n_i'' \equiv \bar{\delta}(\gamma_{i1}n_1' + \gamma_{i2}n_2') \equiv \bar{\delta}(\gamma_{i1}n_1 + \gamma_{i2}n_2) \ (\text{mod } p),$$

$$q_i'' \equiv \bar{\delta}\zeta(\gamma_{i1}q_1' + \gamma_{i2}q_2') \equiv \bar{\delta}\zeta h(\gamma_{i1}q_1 + \gamma_{i2}q_2) \ (\quad \text{``} \quad), \ i = 1, \ 2.$$

Giving the exponents of transformation the following particular values,

$$\delta \equiv h, \ |\gamma| \equiv \begin{vmatrix} \gamma_{11}, & \gamma_{12} \\ -q_2, & q_1 \end{vmatrix} \equiv \zeta \equiv 1 \pmod{p},$$

we have

$$\begin{aligned} h'' &\equiv h' \equiv 1 && \pmod{p}, \\ n_1'' &\equiv \bar{h}(\gamma_{11}n_1 + \gamma_{12}n_2) && (\ \text{“}\ ), \\ q_1'' &\equiv 1 && (\ \text{“}\ ), \\ q_2'' &\equiv 0 && (\ \text{“}\ ). \end{aligned}$$

Let $A = [n_1, n_2, p]$. It is necessary to distinguish whether or not $A \equiv 0 \pmod{p}$. A is invariant as appears from the form of the general relation between the n's and n''s. There is only one set of values for which $A \equiv 0 \pmod{p}$, viz. $n_1 \equiv n_2 \equiv 0 \pmod{p}$.

Suppose $A \equiv 1 \pmod{p}$. We had above

$$\begin{aligned} n_1'' &\equiv h(\gamma_{11}n_1 + \gamma_{12}n_2) \pmod{p}, \\ n_2'' &\equiv h(-q_2n_1 + q_1n_2) \equiv h \begin{vmatrix} q_1 & q_2 \\ n_1 & n_2 \end{vmatrix} \pmod{p}. \end{aligned}$$

In order to leave $q_1'' \equiv 1$, $q_2'' \equiv 0 \pmod{p}$ unaltered, any further transformation is subject to the additional conditions,

$$\gamma_{11} \equiv \delta\bar{\zeta}\bar{h}, \ \gamma_{21} \equiv 0, \ \therefore \gamma_{22} \equiv\equiv 0 \pmod{p}.$$

As before $|\gamma| \equiv \zeta$, $\therefore \gamma_{22} = \zeta^2\bar{\delta}h \pmod{p}$, γ_{12} quite arbitrary.

The results of the transformation are

$$\begin{aligned} n_1''' &\equiv \bar{\delta}(\delta\bar{\zeta}\bar{h}n_1'' + \gamma_{12}n_2'') && \pmod{p}, \\ n_2''' &\equiv \bar{\delta}\gamma_{22}n_2'' && (\ \text{“}\ ), \end{aligned}$$

or,

$$\begin{aligned} n_1''' &\equiv \bar{\zeta}hn_1'' + \bar{\delta}\gamma_{12}h \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} && (\ \text{“}\ ), \\ n_2''' &\equiv \bar{\delta}\gamma_{22}h \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} && (\ \text{“}\ ), \\ &\equiv \zeta^2\bar{\delta}^2h^2 \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} && (\ \text{“}\ ), \end{aligned}$$

If $\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} \equiv 0 \pmod{p}$, then necessarily $n_2''' \equiv 0 \pmod{p}$. It is verified as before that $B \equiv \left[\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix}, p \right]$ is an invariant.

If $A \equiv 1$, and $B \equiv 0 \pmod p$, then $n_1'' \not\equiv 0 \pmod p$. For $n_1'' \equiv h(\gamma_{11}n_1 + \gamma_{12}n_2) \pmod p$, where the γ's satisfy also the congruence

$$\gamma_{11}q_1 + \gamma_{12}q_2 \equiv 1 \pmod p;$$

since $A \equiv 1$, n_1 and n_2 are not both $\equiv 0 \pmod p$, and q_1 and q_2 are not both $\equiv 0 \pmod p$, and hence, since $\begin{vmatrix} q_1 & q_2 \\ n_1 & n_2 \end{vmatrix} \equiv 0 \pmod p$, the value of n_1'' just written is a non-vanishing multiple (mod p) of the congruence $\gamma_{11}q_1 + \gamma_{12}q_2 \equiv 1 \pmod p$, and therefore, $n_1'' \not\equiv 0 \pmod p$.

Since $n_1'' \not\equiv 0 \pmod p$, ζ can always be so chosen that $n_1''' \equiv 1 \pmod p$.

In the case $A \equiv 1$, $B \equiv 0 \pmod p$, there is then but one group.

If $$A \equiv 1,\ B \equiv 1, \text{ i. e. } \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} \not\equiv 0 \pmod p,$$

then $$n_2''' \equiv (\zeta\delta h)^2 \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} \pmod p.$$

If $\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix}$ is a quadratic residue of p, m_2''' can be made $\equiv 1 \pmod p$, while if $\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix}$ is a quadratic non-residue of p, n_2''' can be reduced to any non-residue of p we please. Hence distinction must be made whether

$$C = \left(\frac{\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix}}{p} \right)^* = \pm 1.$$

C is an invariant, as may be verified in the usual way. In either case, m_1''' is made zero by a proper choice of γ_{12} which is perfectly arbitrary.

A, B, C form, therefore, a complete system of invariants whereby are distinguished the following

Groups, Type $(1)(1)(11)$, $p > 3$.

	1.)	2.)	3.)	4.)
$A \equiv$	0,	1,	1,	1,
$B \equiv$	0,	0,	1,	1,
$C =$	$+1$,	$+1$,	$+1$,	-1.

* Legendre's symbol. Consider $\left(\frac{0}{p}\right) = +1$.

48. *Case* $p = 3$.

Formulæ of transformation:

$$h' \equiv \bar{\zeta} |\gamma| h \qquad (\text{mod } 3),$$
$$q'_i \equiv \bar{\delta}\zeta(\gamma_{i1}q_1 + \gamma_{i2}q_2) \quad (\text{ `` }), \; i = 1, 2,$$
$$n'_i \equiv \bar{\delta}\Big[\gamma_{i1}\gamma_{i2}h \left\{ \frac{(\gamma_{i1} - 3)q_1 - (\gamma_{i2} + 3)q_2}{2} \right\} + \gamma_{i1}n_1 + \gamma_{i2}n_2 \Big] \; (\text{mod } 3),$$

or, multiplying by 2 and rejecting multiples of 3,

$$n'_i \equiv \bar{\delta}\{\gamma_{i1}\gamma_{i2}h(\gamma_{i2}q_2 - \gamma_{i1}q_1) + \gamma_{i1}n_1 + \gamma_{i2}n_2\} \; (\text{mod } 3), \; i = 1, 2.$$

The exponent t' has the same value as in the case $p > 3$, and just as in that case it is made zero by giving the arbitrary quantity ε a suitable value.

If $\alpha \not\equiv 0$ (mod 3), then by Fermat's theorem, $\alpha^2 \equiv 1$ and $\bar{\alpha} \equiv \alpha$ (mod 3); throughout the case $p = 3$ we may therefore replace $\bar{\alpha}$ by α.

We first apply a transformation to make $h' \equiv 1$ (mod 3), viz.

$\delta \equiv 1$, $\zeta \equiv h$, $|\gamma| \equiv \begin{vmatrix} 10 \\ 01 \end{vmatrix}$ (mod 3), with the result $n'_i \equiv n_i$, $q'_i \equiv hq_i$, $h' \equiv 1$ (mod 3).

To make $q'' \equiv 1$, $q''_2 \equiv 0$ (mod 3) we now apply the transformation

$$|\gamma| \equiv \begin{vmatrix} 2q_1q_2^2 + q_1, & q_2 \\ -q_2, & q_1 \end{vmatrix} \equiv 1, \; \zeta \equiv 1, \; \delta \equiv h \; (\text{mod } 3),$$

with the results:

$$h'' \equiv 1 \; (\text{mod } 3),$$
$$q''_1 \equiv 1 \; (\text{ `` }),$$
$$q''_2 \equiv 0 \; (\text{ `` }),$$
$$n''_1 \equiv h[(2q_1q_2^2 + q_1)n_1 + q_2n_2] \; (\text{mod } 3),$$
$$n''_2 \equiv q_1^2q_2^2 + h \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} \qquad (\text{ `` })$$

It can be verified as usual that $n''_2 \equiv q_1^2q_2^2 + h \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} \equiv A$ (mod 3), is invariant. We distinguish $A \equiv 0, 1, 2$ (mod 3). If $A \equiv 0$ or 2 (mod 3), γ_{12} can always be so chosen that $n'''_1 \equiv 0$ (mod 3).

If $A \equiv 1$ (mod 3), distinction must be made as to whether or not $n''_1 \equiv 0$ (mod 3). If $n''_1 \equiv 0$, then $n'''_1 \equiv 0$ (mod 3), while if $n''_1 \not\equiv 0$ (mod 3) we make, by a proper choice of ζ, $n'''_1 \equiv 1$ (mod 3).

The expression $A(A+1)n_1''$ is always $\equiv 0$ unless $A\equiv 1 \pmod 3$. In the latter case it is $\equiv 0 \pmod 3$ or not according as n_1'' is so or not. Put

$$B'\equiv A(A+1)h[(2q_1q_2^2+q_1)n_1+q_2n_2] \pmod 3.$$

It can be verified as usual that

$$B\equiv [B', 3] \pmod 3 \text{ is an invariant.*}$$

---

* Since B is perhaps the most difficult invariant to verify, I add the details of the verification. All other invariants which are not readily verified can be treated similarly. If we put $C\equiv h[(2q_1q_2^2+q_1)n_1+q_2n_2]$ (mod 3), and hence $B'\equiv A(A+1)C$ (mod 3), then B is invariant, and indeed congruent zero (mod 3), independently of C, when $A\equiv 0$, (mod 3); but when $A\equiv 1$, then $B'\equiv 2C$ (mod 3), and to show that $B\equiv[B', 3]\equiv[2C, 3]$ is invariant in this case also, it is necessary to show that in this case $[C, 3]$ is itself invariant.

We consider then the case that

$$A\equiv q_1^2q_2^2+h\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2\end{vmatrix}\equiv 1 \pmod 3, \text{ and}$$

prove that for this value of A, $[C, 3]$ is invariant under the most general transformation admissible.

The most general transformation can be made up (Kronecker, Berl. Monatsberichte, 1866, Krazer, Annali di. Mat. Ser. 2, Vol. XII, Königsberger, Ell. Funkt., Vol. II, p. 70) out of the component transformations:

$$\text{I.}\quad \delta\equiv\delta,\quad \zeta\equiv 1,\quad |\gamma|\equiv\begin{vmatrix}1, & 0\\ 0, & 1\end{vmatrix}, \pmod 3,$$

$$\text{II.}\quad \delta\equiv 1,\quad \zeta\equiv\zeta,\quad |\gamma|\equiv\begin{vmatrix}1, & 0\\ 0, & 1\end{vmatrix}, (\;\text{"}\;),$$

$$\text{III.}\quad \delta\equiv 1,\quad \zeta\equiv 1,\quad |\gamma|\equiv\begin{vmatrix}1, & 1\\ 0, & 1\end{vmatrix}, (\;\text{"}\;),$$

$$\text{IV.}\quad \delta\equiv 1,\quad \zeta\equiv 1,\quad |\gamma|\equiv\begin{vmatrix}0, & 1\\ -1, & 0\end{vmatrix}, (\;\text{"}\;),$$

$$\text{V.}\quad \delta\equiv 1,\quad \zeta\equiv 1,\quad |\gamma|\equiv\begin{vmatrix}\gamma_{11}, & 0\\ 0, & 1\end{vmatrix}, (\;\text{"}\;).$$

It appears easily that C is unaltered by transformations I and II. Applying transformation III, and denoting by C' the resulting expression, we have

$$C'\equiv h[\{2(q_1+q_2)q_2^2+q_1+q_2\}\{h(q_2-q_1)+n_1+n_2\}+q_2n_2] \pmod 3,$$

whence $\quad C'\equiv C+h(2q_1q_2^2+q_1)(hq_2-hq_1+n_2) \pmod 3.$

Put $\quad D\equiv h(2q_1q_2^2+q_1)(hq_2-hq_1+n_2)\quad (\;\text{"}\;),$

$\qquad\quad \equiv q_1(q_2^2-1)(q_1-hn_2)\quad (\;\text{"}\;).$

We have then as the result of this transformation $C'\equiv C+D$ (mod 3), and that C may be invariant under this transformation it is necessary and sufficient that $D\equiv 0$ (mod 3). D is evidently $\equiv 0$ (mod 3) if $q_1\equiv 0$, or if $q_2\not\equiv 0$ (mod 3), while if $q_1\not\equiv 0$, $q_2\equiv 0$ (mod 3),

$$D\equiv -q_1-(q_1-hn_2)\equiv -1+hq_1n_2 \pmod 3.$$

But when $q_2\equiv 0$ (mod 3) we have from $A\equiv 1$ (mod 3), $hq_1n_2\equiv 1$ (mod 3) and hence $D\equiv 0$ (mod 3). C is therefore invariant under transformation III when $A\equiv 1$ (mod 3).

Applying transformation IV we have

$$C'\equiv C-hq_1q_2\begin{vmatrix} q_1, & q_2\\ n_1, & n_2\end{vmatrix} \pmod 3.$$

Put

$$E\equiv hq_1q_2\begin{vmatrix} q_1, & q_2\\ n_1, & n_2\end{vmatrix} \pmod 3.$$

Groups, Type (1)(1)(11), $p = 3$.

	1.)	2.)	3.)	4.)
$A \equiv$	0,	1,	1,	2,
$B \equiv$	0,	0,	1,	0.

49. *Case* $p = 2$.

In this case, the generating relations are (cf. No. 44),

$$\begin{aligned} a^2 &= 1, \\ b^2 &= a^m, \\ \left.\begin{aligned} c_i^2 &= a^{n_i} b^{f_i}, \\ c_i b &= a^{q_i} b c_i, \end{aligned}\right\} & \; i = 1,\ 2, \\ c_2 c_1 &= a^t b c_1 c_2, \\ ab &= ba, \\ ac_i &= c_i a, \quad i = 1,\ 2. \end{aligned}$$

Not both q_1 and q_2 may be congruent zero (mod 2), (No. 44).

Formulæ of multiplication (*cf. No.* 44):

$$c_i^\sigma b^\rho = a^{\sigma\rho q} b^\rho c_i^\sigma, \quad i = 1,\ 2,$$

$$c_2^\sigma c_1^\rho = a^{\sigma\rho\left[t + \frac{(\sigma-1)q_2 + (\rho-1)q_1}{2}\right]} . b^{\rho\sigma} . c_1^\rho c_2^\sigma .$$

The following are the new relations consequent upon the given relations by transformation; transforming $c_1^2 = a^{n_1} b^{f_1}$ by c_1 and by c_2, and also $c_2^2 = a^{n_2} b^{f_2}$ by c_1 and by c_2, we reach respectively the following relations:

$$\begin{aligned} a^{f_1 q_1} &= 1, \\ a^{q_1 + f_1 q_2 + m} &= 1, \\ a^{q_2 + f_2 q_1 + m} &= 1, \\ a^{f_2 q_2} &= 1. \end{aligned}$$

That C may be invariant when $A \equiv 1$ (mod 3) we must have then $E \equiv 0$ (mod 3). This is evidently the case if q_1 or $q_2 \equiv 0$ (mod 3), while if $q_1 \not\equiv 0$, $q_2 \not\equiv 0$ (mod 3), we have from $A \equiv 1$, $1 + h \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} \equiv 1$, (mod 3), whence $\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} \equiv 0$ (mod 3), or $E \equiv 0$ (mod 3). Therefore when $A \equiv 1$ (mod 3), C is invariant under transformation IV.

Applying transformation V, we find readily that $C' \equiv \gamma_{11} C$ (mod 3). Since $\gamma_{11} \not\equiv 0$ (mod 3), $[C, 3]$ is invariant under this transformation.

We have now shown that $[C, 3]$ is invariant when $A \equiv 1$, (mod 3), and that, consequently, $B \equiv [B', 3] \equiv [A(A+1)C, 3]$ (mod 3) is an invariant of the type.

Whence (note No. 20),

$$
\begin{aligned}
q_2 + f_2 q_1 + m &\equiv 0 \pmod 2, \\
q_1 + f_1 q_2 + m &\equiv 0 \;(\;\text{"}\;), \\
f_1 q_1 &\equiv 0 \;(\;\text{"}\;), \\
f_2 q_2 &\equiv 0 \;(\;\text{"}\;).
\end{aligned}
$$

From these congruences it appears that

$$
\begin{aligned}
m &\equiv 1 \qquad \pmod 2, \\
f_1 &\equiv 1 + q_1 \;(\;\text{"}\;), \\
f_2 &\equiv 1 + q_2 \;(\;\text{"}\;).
\end{aligned}
$$

Making the following most general choice of generating elements (cf. No. 46),

$$
\begin{aligned}
a' &= a, \\
b' &= a^{\epsilon} b, \\
c_i' &= b^{\beta_i} c_1^{\gamma_{i1}} c_2^{\gamma_{i2}}, \quad i = 1, 2, \quad |\gamma| \not\equiv 0 \pmod 2,
\end{aligned}
$$

and denoting by $G\left[\frac{N}{2}\right]$ the greatest integer in $\frac{N}{2}$ if $\frac{N}{2}$ is positive or zero, and one more than the greatest integer in $\frac{N}{2}$ taken with the positive sign if $\frac{N}{2}$ is negative; and by $R[N]$ the smallest non-negative remainder of N (mod 2), we find as usual the following

Formulæ of transformation:

$$
\begin{aligned}
f_i' &= R[\gamma_{i1}\gamma_{i2} + f_1\gamma_{i1} + f_2\gamma_{i2}]^*, \quad i = 1, 2, \\
n_i' &\equiv \gamma_{i1}\gamma_{i2}\left[t + \frac{(\gamma_{i1} - 1)q_1 + (\gamma_{i2} - 1)q_2}{2} + \gamma_{i1}q_1\right] \\
&\quad + \beta_i[1 + \gamma_{i1}q_1 + \gamma_{i2}q_2] + \gamma_{i1}n_1 + \gamma_{i2}n_2 \\
&\quad + G\left[\frac{\gamma_{i1}\gamma_{i2} + f_1\gamma_{i1} + f_2\gamma_{i2}}{2}\right] \\
&\quad + \epsilon R[\gamma_{i1}\gamma_{i2} + f_1\gamma_{i1} + f_2\gamma_{i2}], \pmod 2, \quad i = 1, 2, \\
q_i' &\equiv \gamma_{i1}q_1 + \gamma_{i2}, \pmod 2, \quad i = 1, 2, \\
t' &\equiv t + \Big\{ \beta_1\gamma_{21} + \beta_2\gamma_{11} + \gamma_{11}\gamma_{21}(\gamma_{22} - \gamma_{12}) \\
&\quad + \frac{\gamma_{11}\gamma_{22}(\gamma_{11} - 1) - \gamma_{12}\gamma_{21}(\gamma_{21} - 1)}{2} \Big\} q_1 \\
&\quad + \left\{ \beta_1\gamma_{22} + \beta_2\gamma_{11} + \frac{\gamma_{11}\gamma_{22}(\gamma_{22} - 1) - \gamma_{12}\gamma_{21}(\gamma_{12} - 1)}{2} \right\} q_2 \\
&\quad + \epsilon + G\left[\frac{1 - |\gamma|}{2}\right], \pmod 2.
\end{aligned}
$$

---

* It is to be noticed particularly that, since $b^2 = a$, the exponents of b in the relations may not be altered by arbitrary multiples of 2, and that the quantities f must therefore be supposed to have their smallest non-negative values (mod 2), and that f_i' must be taken *equal* to the expression given.

We now apply a transformation to make the q's both unity, viz.

$$|\gamma| \equiv \begin{vmatrix} q_1, & q_1+q_2 \\ q_1+q_2, & q_2 \end{vmatrix}, \ \beta_t \equiv 0, \ \varepsilon \equiv 0, \ (\text{mod } 2),$$

and remembering that $\alpha^2 \equiv \alpha$, and $q_1 + q_2 + q_1q_2 \equiv 1$ (mod 2), and supposing the smallest non-negative values of the q's (mod 2) to have been taken originally, we have the results:

$$\begin{aligned} n_1' &\equiv (1+q_2)t + q_1n_1 + (q_1+q_2)n_2, \ (\text{mod } 2), \\ n_2' &\equiv (1+q_1)t + 1 + q_1 + (q_1+q_2)n_1 + q_2n_2, \ (\text{mod } 2), \\ q_1' &= 1; \ f_1' = 0, \\ q_2' &= 1; \ f_2' = 0. \end{aligned}$$

The most general transformation which will leave these values of q_1, q_2 unaltered is

$$|\gamma| \equiv \begin{vmatrix} \gamma_{11}, & 1-\gamma_{11} \\ 1-\gamma_{11}, & \gamma_{11} \end{vmatrix} \ (\text{mod } 2), \ \beta_t, \ \varepsilon \text{ arbitrary.}$$

Neither ε nor β_t has any effect upon the n's when $q_1 \equiv q_2 \equiv 1$, (mod 2). By giving ε a proper value we make $t'' \equiv 0$ (mod 2). The resulting values of n_1n_2 are,

$$\begin{aligned} n_1'' &\equiv \gamma_{11}n_1' + (1-\gamma_{11})n_2' \ (\text{mod } 2), \\ n_2'' &\equiv (1-\gamma_{11})n_1' + \gamma_{11}n_2' \ (\ \text{“}\ ). \end{aligned}$$

Adding and multiplying these values, we have respectively,

$$\begin{aligned} n_1'' + n_2'' &\equiv n_1' + n_2', \ (\text{mod } 2), \\ n_1''n_2'' &\equiv n_1'n_2', \quad (\ \text{“}\ ). \end{aligned}$$

Written in terms of the original parameters, these are then two invariants, viz.

$$\begin{aligned} A &\equiv n_1' + n_2' \equiv (q_1+q_2)t + q_1 + 1 + q_2n_1 + q_1n_2 \ (\text{mod } 2), \\ B &\equiv n_1'n_2' \equiv (1+q_2)tn_1 + (1+q_1)tn_2 + (1+q_2)n_1 + n_1n_2 \ (\text{mod } 2). \end{aligned}$$

Since we have reduced all the other parameters to a single set of values, irrespective of the n's, the invariants A and B distinguish completely the groups.

Groups, Type $(1)(1)(11)$, $p = 2$.

	1.)	2.)	3.)
$A \equiv$	0,	1,	0,
$B \equiv$	0,	0,	1.

50. *Summary of results.*

In the following summary of results there are given first, the generating relations of each type in their general form, then the invariants, and last, all the

existent groups, except the commutative groups, defined as well by values of the invariants, as by generating relations in which the exponents form a set of "reduced" values (No. 25).

GROUPS OF ORDER p^3.

Type $(1)(11)$.

Relations:

$$a^p = 1,$$
$$b_i^p = a^{m_i}, \; i = 1, 2,$$
$$b_2 b_1 = a^n b_1 b_2,$$
$$a b_i = b_i a, \; i = 1, 2.$$

Case $p > 2$.

Invariant: $A = [m_1, m_2, p]$.

Groups: $A = 1, p$.

	1).	2).
$a^p =$	1	1
$b_1^p =$	a	1
$b_2^p =$	1	1

$$\left.\begin{array}{l} b_2 b_1 = a b_1 b_2, \\ a b_i = b_i a, \end{array}\right\} \text{in both groups.}$$

Case $p = 2$.

Invariant: $A \equiv m_1 m_2 \pmod 2$.

Groups: $A \equiv 1, 0 \pmod 2$.

	1).	2).
$a^2 =$	1	1
$b_1^2 =$	a	1
$b_2^2 =$	a	1

$$\left.\begin{array}{l} b_2 b_1 = a b_1 b_2, \\ a b_i = b_i a, \end{array}\right\} \text{in both groups.}$$

GROUPS OF ORDER p^4.

Type $(2)(11)$.

Relations:

$$a^{p^2} = 1,$$
$$b_i^p = a^{m_i}, \quad i = 1, 2,$$
$$b_2 b_1 = a^n b_1 b_2,$$
$$a b_i = b_i a, \quad i = 1, 2.$$

Invariant: $A = [m_1, m_2, p]$.

Groups: $A = p, 1$.

	1).	2).
$a^{p^2} =$	1	1
$b_1^p =$	1	a
$b_2^p =$	1	1

$$\left.\begin{array}{l} b_2 b_1 = a^p b_1 b_2, \\ a b_i = b_i a, \quad i = 1, 2, \end{array}\right\} \text{in both groups.}$$

Type (11)(11).

Relations:

$$\left.\begin{array}{l} a_i^p = 1, \\ b_i^p = a_1^{m_{i1}} a_2^{m_{i2}}, \end{array}\right\} i, = 1, 2,$$

$$b_2 b_1 = a_1^{n_1} a_2^{n_2} b_1 b_2,$$

$$a_2 a_1 = a_1 a_2,$$

$$a_i b_j = b_j a_i, \quad i, j = 1, 2, \text{ independently.}$$

Case $p > 2$.

Invariants:

$$A = [|m|, p],$$

$$B = \left[\begin{vmatrix} n_1, & m_{11}, \\ n_2, & m_{12}, \end{vmatrix}, \begin{vmatrix} n_1, & m_{21} \\ n_2, & m_{22} \end{vmatrix}, p \right];$$

$$C = [m_{11}, m_{12}, m_{21}, m_{22}, p].$$

Groups:

	1).	2).	3).	4).
$A =$	1	p	p	p
$B =$	1	1	p	p
$C =$	1	1	1	p
$a_1^p =$	1	1	1	1
$a_2^p =$	1	1	1	1
$b_1^p =$	a_1	1	1	1
$b_2^p =$	a_2	a_2	a_1	1

$$\left.\begin{array}{l} b_2 b_1 = a_1 b_1 b_2, \\ a_i b_j = b_j a_i, \quad i, j = 1, 2 \\ a_2 a_1 = a_1 a_2 \end{array}\right\} \text{in all these groups.}$$

Case $p = 2$.

Invariants:

$$A \equiv \left[\begin{vmatrix} n_1, & m_{11} \\ n_2, & m_{12} \end{vmatrix}, \begin{vmatrix} n_1, & m_{21} \\ n_2, & m_{22} \end{vmatrix}, 2 \right], \pmod{2},$$

$$B \equiv \begin{vmatrix} m_{11}, & n_1 + n_2 \\ m_{12}, & n_1 \end{vmatrix} \cdot \begin{vmatrix} m_{21}, & n_1 + n_2 \\ m_{22}, & n_1 \end{vmatrix} \cdot (A+1), \pmod{2},$$

$$C \equiv \begin{vmatrix} n_1, & m_{11} \\ n_2, & m_{12} \end{vmatrix} \cdot \begin{vmatrix} n_1, & m_{21} \\ n_2, & m_{22} \end{vmatrix} + \begin{vmatrix} m_{11}, & m_{12} \\ m_{21}, & m_{22} \end{vmatrix}, \pmod{2}.$$

Groups:

	1).	2).	3).	4).
$A \equiv$	0	1	1	0
$B \equiv$	1	0	0	0
$C \equiv$	0	1	0	0
$a_1^2 =$	1	1	1	1
$a_2^2 =$	1	1	1	1
$b_1^2 =$	a_1	a_1	1	1
$b_2^2 =$	a_1	a_2	a_2	1

$$\left.\begin{aligned} b_2 b_1 &= a_1 b_1 b_2, \\ a_i b_j &= b_j a_i, \quad i, j = 1, 2, \\ a_1 a_2 &= a_2, a_1, \end{aligned}\right\} \text{in all these groups.}$$

Type $(1)(1)(11)$.

Relations:

$$\begin{aligned} a^p &= 1, \\ b^p &= a^m, \\ \left.\begin{aligned} c_i^p &= a^{n_i} b^{f_i}, \\ c_i b &= a^{q_i} b c_i, \end{aligned}\right\} & \; i = 1, 2, \\ c_2 c_1 &= a^t b^h c_1 c_2, \\ ab &= ba, \\ ac_i &= c_i a, \quad i = 1, 2. \end{aligned}$$

Case $p > 3$.

Invariants: $A \equiv [n_1, n_2, p], \pmod{p}$,

$$B \equiv \left[\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix}, p \right], \pmod{p},$$

$$C = \left(\frac{\begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix}}{p} \right).$$

Groups:

	1).	2).	3).	4).
$A \equiv$	0	1	1	1
$B \equiv$	0	0	1	1
$C =$	$+1$	$+1$	$+1$	-1
$a^p =$	1	1	1	1
$b^p =$	1	1	1	1
$c_1^p =$	1	a	1	1
$c_2^p =$	1	1	a	a^N

(N any quadratic non-residue of p),

$$\left.\begin{aligned} c_1b &= abc_1, \\ c_2b &= bc_2, \\ c_2c_1 &= bc_1c_2, \\ ab &= ba, \\ ac_i &= c_ia, \quad i = 1,\ 2 \end{aligned}\right\} \text{ in all these groups.}$$

Case $p = 3$.

Invariants: $A \equiv q_1^2q_2^2 + h \begin{vmatrix} q_1, & q_2 \\ n_1, & n_2 \end{vmatrix} \pmod 3$,

$$B \equiv [A(A+1)h\{(2q_1q_2^2 + q_1)n_1 + q_2n_2\},\ 3] \pmod 3.$$

Groups:

	1).	2).	3).	4).
$A \equiv$	0	1	1	2
$B \equiv$	0	0	1	0
$a^3 =$	1	1	1	1
$b^3 =$	1	1	1	1
$c_1^3 =$	1	1	a	1
$c_2^3 =$	1	a	a	a^2

$$\left.\begin{aligned} c_1b &= abc_1, \\ c_2b &= bc_2, \\ c_2c_1 &= bc_1c_2, \\ ab &= ba, \\ ac_i &= c_ia. \end{aligned}\right\} \text{ in all these groups.}$$

Case $p = 2$.

Invariants:

$$A \equiv (q_1 + q_2)t + q_1 + 1 + q_2n_1 + q_1n_2 \pmod 2,$$

$$B \equiv (1 + q_2)tn_1 + (1 + q_1)tn_2 + (1 + q_2)n_1 + n_1n_2 \pmod 2.$$

Groups:

	1).	2).	3).
$A \equiv$	0	1	0
$B \equiv$	0	0	1
$a^2 =$	1	1	1
$b^2 =$	a	a	a
$c_1^2 =$	1	a	a
$c_2^2 =$	1	1	a

$$\left.\begin{array}{ll} c_i b = abc_i, & i = 1, 2, \\ c_2 c_1 = bc_1 c_2, & \\ ab = ba, & \\ ac_i = c_i a, & i = 1, 2, \end{array}\right\} \text{in all these groups.}$$

§3.

51. *Groups of order* 16.

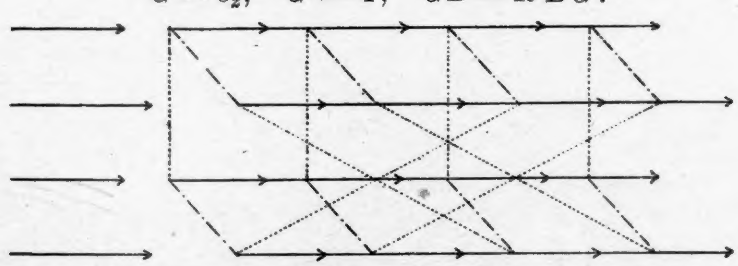
I add a table of the "color groups,"* presenting graphically, in Cayley's method, the non-commutative groups of order 16 as regular transitive groups of substitutions operating on sixteen letters.

Each group can be generated by three, or fewer, elements properly chosen. To these we shall assign the colors red, green and black, and shall denote them by the letters R, G, B, and in the figures by the lines ————, ..., —·—·—·—·—·—, respectively. We shall give, for each group, the new generating elements expressed in terms of those of the preceding paragraph, then the relations which the new generating elements satisfy, and last, the color-diagrams.

Type (2)(11).

Group 1).

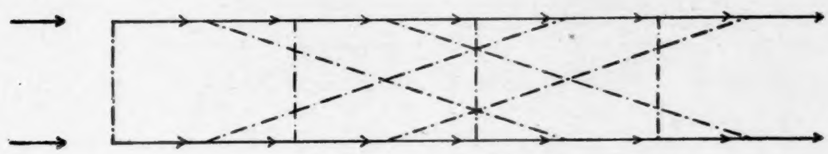
$$R = a, \quad R^4 = 1, \quad RB = BR,$$
$$B = b_1, \quad B^2 = 1, \quad RG = GR,$$
$$G = b_2, \quad G^2 = 1, \quad GB = R^2BG.$$



* Cayley, Am. J. Math., Vol. XI, p. 139.

Group 2).

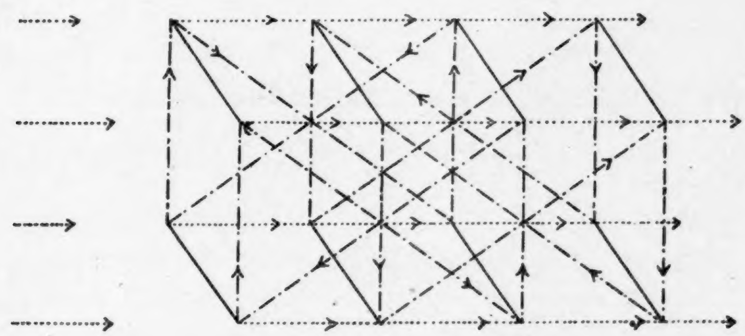
$$R = b_1, \quad R^8 = 1,$$
$$B = b_2, \quad B^2 = 1, \quad BR = R^5 B.$$



Type (11)(11).

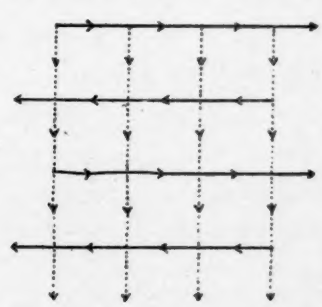
Group 1).

$$R = a_2, \quad G^4 = 1, \quad G^2 = B^2,$$
$$G = b_1, \quad B^4 = 1, \quad BG = G^3 B,$$
$$B = b_2, \quad R^2 = 1, \quad RG = GR, \quad RB = BR.$$



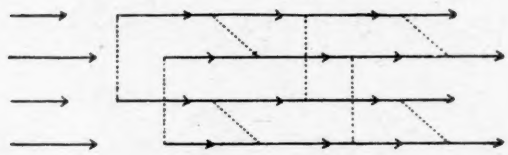
Group 2).

$$R = b_1, \quad R^4 = 1,$$
$$G = b_2, \quad G^4 = 1, \quad GR = R^3 G.$$



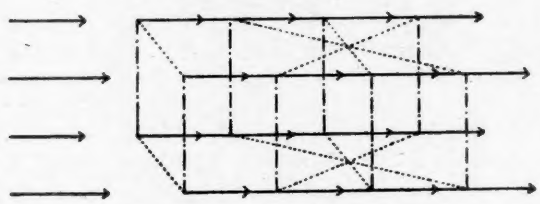
Group 3).

$$R = b_2 b_1, \quad R^4 = 1,$$
$$G = b_1, \quad G^2 = 1, \quad R^2 G = G R^2.$$



Group 4).

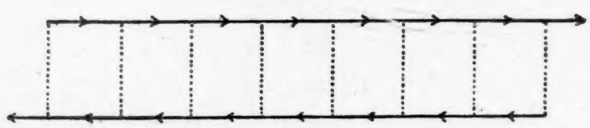
$$B = a_2, \quad R^4 = 1, \quad GB = BG,$$
$$R = b_2 b_1, \quad B^2 = 1, \quad RB = BR,$$
$$G = b_1, \quad G^2 = 1, \quad GR = R^3 G.$$



Type (1)(1)(11).

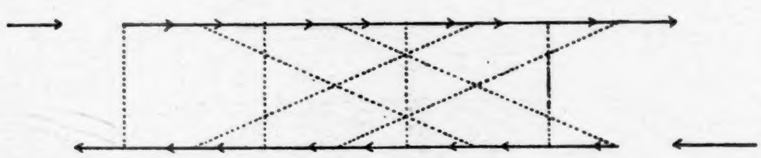
Group 1).

$$R = c_2 c_1, \quad R^8 = 1,$$
$$G = c_2, \quad G^2 = 1, \quad RG = GR^7.$$



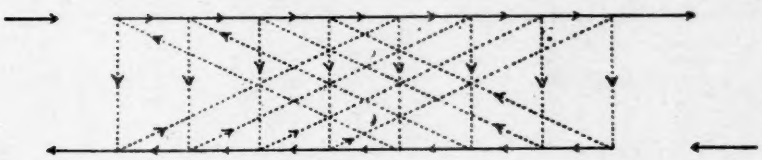
Group 2).

$$R = c_2 c_1, \quad R^8 = 1,$$
$$G = c_2, \quad G^2 = 1, \quad RG = GR^3.$$



Group 3).

$$R = c_2c_1,\quad R^8 = 1,\quad G^2 = R^4,$$
$$G = c_2,\quad G^4 = 1,\quad GR = R^7G.$$



52. It is of interest to compare with these results the various particular groups of order 16 already known. I shall mention here only the most important groups of order 16 previously given, viz. those contained in Cayley's list of groups of substitutions on eight letters.* To determine whether or not a given group of order 16 is expressible as a group of substitutions on eight letters, and if so, to find the group of substitutions, we use the method given by Dyck.† It is, in brief, this:

If a group G, of order $N = \mu . \nu$, contains a subgroup of order μ, the elements of G can be distributed into the following "systems of imprimitivity":

$$1, A_2, \ldots . A_\mu \,|\, B_2, A_2B_2, \ldots . A_\mu B_2 \,|\, \ldots\ldots\ldots . \,|\, B_\nu, A_2B_\nu, \ldots . A_\mu B_\nu .$$

Call these systems respectively $C_1, C_2, \ldots . C_\nu$. By the definition of imprimitivity it follows that when the elements of the group G are multiplied throughout by any element of G, there results a substitution on $C_1, C_2, \ldots . C_\nu$. Multiplying the elements of G successively by its N elements, there result N substitutions on $C_1, C_2, \ldots . C_\nu$, forming a group G'. Some of these substitutions may however be identical. Dyck shows that the necessary and sufficient condition that the group G' (of substitutions on ν letters) be actually of order N and holoedrically isomorphic with the group G, (identical with G in the sense of this paper), is that the subgroup $1, A_2, \ldots . A_\mu$, used in the determination of G', shall neither be a self-conjugate subgroup of G nor contain any self-conjugate subgroup of G.

In our case, $N = 16$, $\nu = 8$, $\mu = 2$, and every subgroup of order 2, not a self-conjugate subgroup of G, leads to a group of order 16 of substitutions on eight letters. Making examination in this way of the non-commutative groups

* Quar. J. Math. No. 98, 1891, p. 137.

† Math. Annalen, Vol. XXII, p. 90.

of order 16 enumerated above, we find that the following of them can be expressed as transitive groups of substitutions on eight letters:

Group.		*Number in Cayley's list.**
(2)(11),	1), ...	20,
	2), ...	23,
(11)(11),	3), ...	22,
	4), ...	21,
(1)(1)(11),	1), ...	v. below,
	2), ...	"

The group (1)(1)(11), 1), expressed as a transitive group of substitutions operating on eight letters, is generated by the substitutions

$$(bh)(cg)(df)$$

and

$$(abcdefgh),$$

while the group (1)(1)(11), 2), similarly expressed, is generated by the substitutions

$$(bd)(cg)(fh)$$

and

$$(abcdefgh).$$

I have not been able to identify these two groups with any of the groups of Cayley's list.

§4.

53. *Type* $(\kappa)(11 \ldots . 1)$, s units in the second parenthesis.

The groups of this type are generated by elements a, b_i, $(i = 1, 2, \ldots . s,)$ satisfying the relations:

$$\begin{array}{ll} a^{p^\kappa} = 1, & \\ b_i^h = a^{m_i} & i = 1, 2, \ldots . s \\ b_j b_i = a^{n_{ij}} b_i b_j, & i, j = 1, 2 \ldots . s, \text{ independently,} \\ a b_i = b_i a, & i = 1, 2 \ldots . s. \end{array}$$

From the third relation we have

$$n_{ij} \equiv - n_{ji}, \; n_{ii} \equiv 0, \; (\text{mod } p^\kappa).$$

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* Quar. J. Math. 1891, p. 140.

Formulæ of multiplication:

$$b_j^{\beta_j} b_i^{\beta_i} = a^{\beta_j \beta_i n_{ij}} b_i^{\beta_i} b_j^{\beta_j},$$

$$(a^{a} b_1^{\beta_1} b_2^{\beta_2} \ldots\ldots b_s^{\beta_s})^q = a^{q a + \frac{q(q-1)}{2} \sum\limits_{j=i+1}^{s} \sum\limits_{i=1}^{s-1} \beta_i \beta_j n_{ij}} \cdot b_1^{q\beta_1} b_2^{q\beta_2} \ldots\ldots b_s^{q\beta_s},$$

$$b_1^{\beta_{l1}} b_2^{\beta_{l2}} \ldots\ldots b_s^{\beta_{ls}} \cdot b_1^{\beta_{\kappa 1}} b_2^{\beta_{\kappa 2}} \ldots\ldots b_s^{\beta_{\kappa s}} = a^{\sum\limits_{j=i+1}^{s} \sum\limits_{i=1}^{s-1} \beta_{\kappa i} \beta_{lj} n_{ij}} \cdot b_1^{\beta_{l1} + \beta_{\kappa 1}} b_2^{\beta_{l2} + \beta_{\kappa 2}} \ldots\ldots b_s^{\beta_{ls} + \beta_{\kappa s}},$$

$$b_1^{\beta_{l1}} b_2^{\beta_{l2}} \ldots\ldots b_s^{\beta_{ls}} \cdot b_s^{\beta_{hs}} b_{s-1}^{\beta_{h,s-1}} \ldots\ldots b_2^{\beta_{h2}} b_1^{\beta_{h1}} = a^{\sum\limits_{j=\kappa+1}^{s} \sum\limits_{\kappa=1}^{s-1} (\beta_{lj} + \beta_{hj}) \beta_{h\kappa} n_{\kappa j}} \cdot b_1^{\beta_{h1} + \beta_{l1}} b_2^{\beta_{h2} + \beta_{l2}} \ldots\ldots b_s^{\beta_{hs} + \beta_{ls}},$$

$$b_s^{\beta_{is}} b_{s-1}^{\beta_{i,s-1}} \ldots\ldots b_2^{\beta_{i2}} b_1^{\beta_{i1}} \cdot b_1^{\beta_{j1}} b_2^{\beta_{j2}} \ldots\ldots b_s^{\beta_{js}} = a^{\sum\limits_{\kappa=l+1}^{s} \sum\limits_{l=1}^{s-1} \beta_{i\kappa} (\beta_{jl} + \beta_{il}) n_{l\kappa}} \cdot b_1^{\beta_{j1} + \beta_{i1}} b_2^{\beta_{j2} + \beta_{i2}} \ldots\ldots b_s^{\beta_{js} + \beta_{is}}.$$

By transforming the second relation given by b_j we find that

$$a^{p n_{ij}} b_i^p = a^{m_i}, \text{ and hence}$$

$$a^{p n_{ij}} = 1.$$

From this it follows (v. note to No. 20) that $p_{n_{ij}} \equiv 0 \pmod{p^{\kappa}}$ or $n_{ij} \equiv 0 \pmod{p^{\kappa-1}}$ for all values of i and j. This restriction upon the n's is necessary in order that the group generated by the elements satisfying the relations given may be of the type in question. Supposing the n's chosen subject to this restriction, no new relations arise from transforming all of the generating relations by each of the generating elements, and the only consequences of the given relations are, therefore, their products and powers.

54. We next determine the condition that no element other than a and its powers be commutative with every element of the group. Suppose

$$g = a^{a} b_1^{\gamma_1} \ldots\ldots b_s^{\gamma_s}$$

were commutative with every element of the group. To this end it is necessary and sufficient that g be commutative severally with $b_1, b_2, \ldots\ldots b_s$. For g is certainly commutative with a, and every element of the group can be expressed as a product of a and b_i's. I. e., that g be commutative with every element of the group it is necessary and sufficient that

$$g b_l = b_l g,$$

or $$a^{a} b_1^{\gamma_1} \ldots\ldots b_s^{\gamma_s} b_l = b_l \cdot a^{a} b_1^{\gamma_1} \ldots\ldots b_s^{\gamma_s}, \quad l = 1, 2 \ldots\ldots s.$$

Whence, reducing and applying multiplication formulæ,

$$a^{\sum\limits_{j=l+1}^{s} \gamma_j n_{lj}} = a^{\sum\limits_{j=1}^{l-1} \gamma_j n_{lj}}, \quad l = 1, 2 \ldots\ldots s.$$

From this,

$$\sum_{j=l+1}^{s} \gamma_j n_{lj} \equiv \sum_{j=1}^{l-1} \gamma_j n_{jl}, \text{ (mod. } p^\kappa), \quad l = 1, 2 \ldots . s,$$

or since $n_{lj} \equiv - n_{jl}$, $n_{jj} \equiv 0$, (mod p^κ),

$$\sum_{j=1}^{s} \gamma_j n_{jl} \equiv 0 \text{ (mod } p^\kappa), \quad l = 1, 2, \ldots . s.$$

Every set of γ's satisfying this system of s congruences determines an element $a^\alpha b_1^{\gamma_1} b_2^{\gamma_2} \ldots . b_s^{\gamma_s}$, which is commutative with every element of the group. But, by hypothesis, no elements other than powers of a are commutative with every element of the group; hence the only admissible values of γ are $\gamma_j \equiv 0$ (mod p), $j = 1, 2, \ldots . s$. The n's must, therefore, be further restricted to such values that the system of congruences above can be satisfied only by $\gamma_j \equiv 0$ (mod p), $j = 1, 2, \ldots . s$.

55. Frobenius* has shown that there is a one-to-one correspondence between the system of congruences under consideration and those of the system

$$\sigma_j \gamma'_j \equiv 0 \text{ (mod } p^\kappa), \quad j = 1, 2, \ldots . s,$$

where σ_j is a quantity defined as follows: Given a square array of m^2 integers. Let d_λ be the positive greatest common divisor of all the determinants of order λ that can be formed by striking out $m - \lambda$ rows and $m - \lambda$ columns of this array. In case the value of every determinant of order λ should be zero, d_λ is taken to be zero also. Since every determinant of order λ is a linear homogeneous function of determinants of order $\lambda - 1$, $e_\lambda = \dfrac{d_\lambda}{d_{\lambda-1}}$ is also an integer. $\Big($We fix that $e_1 = d_1$, and if $e_\mu = \dfrac{0}{0}$, then $e_\mu = 0.\Big)$ The integer e_λ, so defined, is called the λ^{th} *elementary divisor* of the given array, and σ_λ is defined by the following equation:

$$\sigma_\lambda = [e_\lambda, p^\kappa].$$

Frobenius shows† that $\dfrac{\sigma_\lambda}{\sigma_{\lambda-1}}$ is also an integer.

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* Crelle's Journal, Vol. 86, pp. 191-2. See also Smith, Phil. Trans. 1861, p. 293. The notation used above is that of Frobenius.

† *Loc. cit.*, pp. 189-90.

That the congruences

$$\sum_{j=1}^{s} \gamma_j n_{jl} \equiv 0 \pmod{p^\kappa}, \quad l = 1, 2, \ldots . s,$$

be satisfied only by $\gamma_j \equiv 0 \pmod{p^\kappa}$, $j = 1, 2, \ldots . s$, it is necessary and sufficient that the congruences

$$\sigma_j \gamma_j' \equiv 0 \pmod{p^\kappa}, \quad j = 1, 2, \ldots . s$$

(σ_j being now defined by the array of integers n_{ij}), be satisfied only by

$$\gamma_j' \equiv 0 \pmod{p}, \quad j = 1, 2, \ldots . s.$$

And to this end it is necessary and sufficient that $\sigma_j \not\equiv 0 \pmod{p^\kappa}$, $j = 1, 2, \ldots . s$, or, since $\dfrac{\sigma_\lambda}{\sigma_{\lambda-1}}$ is an integer, that

$$\sigma_s \not\equiv 0 \pmod{p^\kappa}.$$

But

$$\sigma_s = [e_s, p^\kappa] = \left[\frac{d_s}{d_{s-1}}, p^\kappa\right] = \left[\frac{|n|}{d_{s-1}}, p^\kappa\right].$$

Hence, that $\sigma_s \not\equiv 0 \pmod{p^\kappa}$ it is necessary that $|n|$ be not *absolutely* zero in value, although the congruence $|n| \equiv 0 \pmod{p^\kappa}$ may hold. This, then, is the condition that no element but a and its powers be commutative with every element of the group.

Further, $|n|$ is a skew symmetric determinant, and not being zero, it must be of *even* order, $s = 2\rho$.* Hence follows the theorem:

The last parenthesis in the type symbol must always contain an even number of integers.

56. We make now the most general choice of generating elements, a', b_i', for the group G, viz:

$$a' = a^\delta,$$
$$b_i' = a_1^{\alpha_i} b_2^{\beta_{i1}} b_1^{\beta_{i2}} \ldots . b_s^{\beta_{is}}, \quad i = 1, 2, \ldots . s, \; s = 2\rho.$$

By the methods already explained (Nos. 22, 23), it appears that the necessary and sufficient conditions that the elements a', b_i' generate the group G itself are

$$\delta \not\equiv 0, \quad |\beta| \not\equiv 0 \pmod{p},$$

α_i being arbitrary.

* Baltzer, Determinanten, §§5, 8.

The cases $p > 2$ and $p = 2$ must now be distinguished.

Formulæ of Transformation, $p > 2$.

$$\delta\left(pa_i + \sum_{j=1}^{s} \beta_{ij} m_j\right) \equiv m'_i \;(\text{mod } p^{\kappa}),\quad i = 1, 2, \ldots\ldots s,$$

$$\delta \sum_{h=1}^{s} \sum_{l=1}^{s} \beta_{il}\beta_{jh} n_{lh} \equiv n'_{ij} \;(\text{mod } p^{\kappa}),\quad \begin{array}{l} i = 1, 2, \ldots\ldots s-1, \\ j = i+1, \ldots\ldots s. \end{array}$$

It appears from the form of the expression for m'_i, together with the fact that the m's are expressible in terms of the m's by like formulæ, that

$$A = [m_1, m_2, \ldots\ldots m_s, p]$$

is an invariant.

Frobenius* has shown that a system of integers β_{ij}, of determinant unity, can be chosen such that

$$\left.\begin{array}{l} n'_{i,\,\rho+i} \equiv \sigma_{2i} \;(\text{mod } p^{\kappa}) \\ n'_{\rho+i,\,i} \equiv -\,\sigma_{2i} \;(\text{mod } p^{\kappa}) \end{array}\right\} i = 1, 2, \ldots\ldots \rho,$$

$$n'_{ij} \equiv 0 \quad (\text{mod } p^{\kappa}) \text{ for all other values of } i \text{ and } j.$$

In view of the restrictions upon the integers n_{ij} already found, we have $e_1 = rp^{\kappa-1}$, $r \not\equiv 0 \;(\text{mod } p)$, $\sigma_1 = p^{\kappa-1}$. Since $\frac{\sigma_\lambda}{\sigma_{\lambda-1}}$ is an integer and $\sigma_{2\rho} \not\equiv 0 \;(\text{mod } p^{\kappa})$, $\sigma_j = p^{\kappa-1}$ for all values of j.

We have, then,

$$n'_{i,\,\rho+1} \equiv -\,n_{\rho+i,\,i} \equiv p^{\kappa-1} \;(\text{mod } p^{\kappa}),\; i = 1, 2, \ldots\ldots \rho,$$

$$n'_{ij} \equiv 0 \;(\text{mod } p^{\kappa}) \text{ for all other values of } i \text{ and } j.$$

We apply now the most general transformation, leaving the n''s unaltered. To this end the β's and δ must be so chosen that

$$\delta \sum_{h=1}^{s} \sum_{l=1}^{s} \beta_{il}\beta_{jh} n'_{lh} \equiv n'_{ij} \;(\text{mod } p^{\kappa}) \begin{array}{l} i = 1, 2, \ldots\ldots s-1, \\ j = i+1, \ldots\ldots s, \end{array}$$

*Crelle's Journal, Vol. 86, pp. 146–191; in particular, pp. 165–168, 187–191. There is a slight difference in form between the result given above and Frobenius' result. If, in the bilinear form $\sum_{\alpha,\beta} n_{\alpha\beta} x_\alpha y_\beta$, used by Frobenius, $y_{2\kappa}$, $x_{2\kappa}$, $y_{2\kappa-1}$, $x_{2\kappa-1}$, be replaced respectively by $y_{\rho+\kappa}$, $x_{\rho+\kappa}$, y_κ, x_κ, the result is reached in the form above.

or, substituting the values of n', that

$$\delta \sum_{l=1}^{\rho} (\beta_{il}\beta_{j,\,\rho+l} - \beta_{i,\,\rho+l}\beta_{jl})\,p^{\kappa-1} \equiv p^{\kappa-1} \pmod{p^{\kappa}} \text{ if } i = j - \rho,$$

$$\equiv -\,p^{\kappa-1} \pmod{p^{\kappa}} \text{ if } i = j + \rho,$$

$$\equiv 0 \pmod{p^{\kappa}} \text{ in all other cases,}$$

i. e., that*

$$\sum_{l=1}^{\rho} (\beta_{il}\beta_{j,\,\rho+l} - \beta_{i,\,\rho+l},\ \beta_{jl}) \equiv \delta \pmod{p}, \text{ if } i = j - \rho,$$

$$\equiv -\,\delta \pmod{p}, \text{ if } i = j + \rho,$$

$$\equiv 0 \pmod{p} \text{ in all other cases.}$$

57. We have already found an invariant A. When $A = p$, then

$$m_i \equiv 0 \pmod{p},\ i = 1,\ 2, \ldots .\ s.$$

In this case it is always possible, by a proper choice of α_i, to make

$$m_i \equiv 0 \pmod{p^{\kappa}},\ i = 1,\ 2, \ldots .\ s.$$

All the sets of values of m_i for which $A = p$ are then reducible to the set just written; i. e., there is only one group for which $A = p$.

In case $A = 1$, it is possible by a proper choice of α_i always to make $0 \leqq m_i < p$. We suppose this done. The following are particular sets of values of δ, β_{ij}, which satisfy the conditions at the end of No. 56:

$$|\beta| \equiv \left|\begin{array}{cccccc|cccccc}
1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \beta_{ii} & \cdots & 0 & 0 & \cdots & \beta_{i,\,\rho+i} & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \beta_{\rho+i,\,i} & \cdots & 0 & 0 & 0 & \cdots & \beta_{\rho+i,\,\rho+i} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1
\end{array}\right| \equiv 1 \pmod{p^{\kappa}}, \qquad \delta \equiv 1 \pmod{p^{\kappa}},$$

*Cf. Clebsch u. Gordan, Abel'sche Funktionen, p. 300.

provided that
$$\begin{vmatrix} \beta_{ii}, & \beta_{i,\,\rho+i} \\ \beta_{\rho+i,\,i}, & \beta_{\rho+i,\,\rho+i} \end{vmatrix} \equiv 1 \pmod{p^\kappa}.$$

We shall call the transformation in which β_{ij} and δ have these values *the transformation A_i*. Similarly, *the transformation B_{ij}* shall be the transformation in which δ and the β's have the following values:

$$|\beta| \equiv \left|\begin{array}{l|l}
1\ \ 0 \ldots 0 \ldots\ldots 0 \ldots\ldots 0 & 0 \ldots\ldots\ldots\ldots\ldots 0 \\
0\ \ 1 \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ \ \beta_{ii} \ldots 0 \ldots\ \beta \ \ldots 0 & \ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ 0 \ldots 1 \ldots\ldots 0 \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ 1 \ldots 0 \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ \beta_{ji} \ldots 0 \ldots 0 \ldots \beta_{jj} \ldots 0 & \ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ 1 \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
0\ \ 0 \ldots 0 \ldots 0 \ldots 0 \ldots 0 \ldots 1 & 0 \ldots\ldots\ldots\ldots\ldots 0 \\
\hline
0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 & 1\ \ 0 \ldots\ldots\ldots\ldots 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots & 0\ \ 1 \ldots\ldots\ldots\ldots 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots & \quad \beta_{\rho+i,\,\rho+i}\ \beta_{\rho+i,\,\rho+j} \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots & \quad\ \vdots \qquad\qquad \vdots \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ \beta_{\rho+j,\,\rho+i}\ \beta_{\rho+j,\,\rho+j} \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 & 0\ \ 0 \ldots\ldots\ldots\ldots 1
\end{array}\right| \equiv 1 \pmod{p^\kappa}, \quad \delta \equiv 1 \pmod{p^\kappa}$$

provided that

$$\begin{vmatrix} \beta_{ii}, & \beta_{ij} \\ \beta_{ji}, & \beta_{jj} \end{vmatrix} \cdot \begin{vmatrix} \beta_{\rho+i,\,\rho+i}, & \beta_{\rho+i,\,\rho+j} \\ \beta_{\rho+j,\,\rho+i}, & \beta_{\rho+j,\,\rho+j} \end{vmatrix} \equiv 1 \pmod{p^\kappa}.$$

Therefore, any set of values satisfying the condition $\begin{vmatrix} \beta_{ii}, & \beta_{ij} \\ \beta_{ji}, & \beta_{jj} \end{vmatrix} \not\equiv 0 \pmod{p}$ is admissible.

58. In case m_i and $m_{\rho+i}$ are not both $\equiv 0 \pmod{p}$, it is always possible (remembering that we have already made every $m < p$), by the transformation A_i to make $m'_i \equiv 1$, $m'_{\rho+i} \equiv 0 \pmod{p^\kappa}$. The m's have now been reduced to such values that

$$m'_{\rho+i} \equiv 0 \pmod{p^\kappa}, \; i = 1, 2, \ldots . \rho,$$

and at least one m'_i is $\not\equiv 0 \pmod{p}$ since $A = 1$.

If m'_i and m'_j, $(j > i)$ are not both $\equiv 0 \pmod p$, it is always possible by the transformation B_{ij}, to make

$$m''_i \equiv 1 \pmod{p^\kappa},$$
$$m''_j \equiv 0 \;(\;\text{''}\;).$$

This transformation leaves all other m's unaltered. By successive applications of B_{ij}, the m's are reduced to

$$m_1 \equiv 1 \pmod{p^\kappa},\quad m_j \equiv 0 \pmod{p^\kappa},\ j = 2,\ 3, \ldots .\ 2\rho,$$

i. e. all the sets of values of m_i for which $A = 1$ can be reduced to the above set, and there is hence but a single group when $A = 1$.

Groups, Type $(\kappa)(11 \ldots . 1)$, $p > 2$.

$A =$	1	p	
$a^{p^\kappa} =$	1	1	
$b_1^p =$	a	1	
$b_j^p =$	1	1	$j = 2,\ 3, \ldots .\ 2\rho.$

In both groups,

$$b_jb_i = a^{p^{\kappa-1}}b_ib_j,\ \text{if } i = j - \rho$$
$$= a^{-p^{\kappa-1}}b_ib_j,\ \text{if } i = j + \rho,$$
$$= b_ib_j \text{ in all other cases,}$$
$$i, j = 1,\ 2, \ldots .\ 2\rho, \text{ independently,}$$
$$ab_i = b_ia,\quad i = 1,\ 2, \ldots .\ 2\rho.$$

59. *Case* $p = 2$.

Formulæ of Transformation:

$$m'_i \equiv 2\alpha_i + \sum_{j=l+1}^{s}\sum_{l=1}^{s-1}\beta_{il}\beta_{ij}n_{lj} + \sum_{j=1}^{s}\beta_{ij}m_j \pmod{2^\kappa},$$

$$n'_{ij} \equiv \sum_{\kappa=1}^{s}\sum_{l=1}^{s}\beta_{il}\beta_{j\kappa}n_{l\kappa} \pmod{2^\kappa},\ i = 1,\ 2, \ldots .\ s-1,\ j = i+1, \ldots .\ s.$$

Since n_{ij} is divisible by $2^{\kappa-1}$ (No. 53), it follows that, when $\kappa > 1$, the second term in the expression above for m_i is a multiple of 2 and can therefore be taken together with the term $2\alpha_i$, in which α_i is a perfectly arbitrary quantity. In this way the formulæ of transformation become precisely those of the case $p > 2$. It is therefore not necessary to consider separately the case $p = 2$ unless $\kappa = 1$. *The groups of type* $(\kappa)(11 \ldots . 1)$ *are those of No.* 58, *irrespective of the value of* p.

60. We consider, then,

Type (1)(11 .... 1), $p=2$.

Formulæ of Transformation:

$$m'_i \equiv \sum_{j=l+1}^{s} \sum_{l=1}^{s-1} \beta_{il}\beta_{ij} n_{lj} + \sum_{j=1}^{s} \beta_{ij} m_j \pmod 2, \; i = 1, 2, \ldots . s,$$

$$n'_{ij} \equiv \sum_{\kappa=1}^{s} \sum_{l=1}^{s} \beta_{il}\beta_{j\kappa} n_{l\kappa} \pmod 2, \; i = 1, 2, \ldots . s-1, \; j = i+1, \ldots . s.$$

As in No. 56, we apply first a transformation making

$$n_{i,\,\rho+i} \equiv n_{\rho+i,\,i} \equiv 1 \pmod 2, \; i = 1, 2, \ldots . \rho,$$
$$n_{ij} \equiv 0 \pmod 2 \text{ for all other values of } i \text{ and } j.$$

The transformation A_i has the following results:

$$m'_i \equiv \beta_{ii}\beta_{i,\,\rho+i} + \beta_{ii} m_i + \beta_{i,\,\rho+i} m_{\rho+i} \pmod 2,$$
$$m'_{\rho+i} \equiv \beta_{\rho+i,\,i}\beta_{\rho+i,\,\rho+i} + \beta_{\rho+i,\,i} m_i + \beta_{\rho+i,\,\rho+i} m_{\rho+i} \pmod 2,$$
$$m'_j \equiv m_j \pmod 2, \text{ if } j \neq i \text{ or } \rho+i.$$

By means of this transformation m'_i and $m'_{\rho+i}$ can both be made $\equiv 0 \pmod 2$ except when m_i and $m_{\rho+i}$ are both congruent unity (mod 2), when necessarily also

$$m'_i \equiv m'_{\rho+i} \equiv 1 \pmod 2.$$

The transformation B_{ij} has the following results:

$$m'_i \equiv \beta_{ii} m_i + \beta_{ij} m_j \pmod 2,$$
$$m'_j \equiv \beta_{ji} m_i + \beta_{jj} m_j \pmod 2,$$
$$m'_{\rho+i} \equiv \beta_{\rho+i,\,\rho+i},\, m_{\rho+i} + \beta_{\rho+i,\,\rho+j} m_{\rho+j} \pmod 2,$$
$$m'_{\rho+j} \equiv \beta_{\rho+j,\,\rho+i},\, m_{\rho+i} + \beta_{\rho+j,\,\rho+j} m_{\rho+j} \pmod 2,$$
$$m'_\kappa \equiv m_\kappa \pmod 2 \text{ for all other values of } \kappa.$$

Provided at least one of the numbers m_i, m_j, is $\equiv 1 \pmod 2$, we can, by this transformation, make

$$m'_i \equiv 1 \pmod 2,$$
$$m'_j \equiv 0 \;(\;\text{"}\;).$$

By successive applications of the transformations A_i and B_{ij}, any possible set of m's can be reduced to one of the two following:

m_1,	m_2,	....	m_ρ,	$m_{\rho+1}$,	$m_{\rho+2}$,	....	$m_{2\rho}$,
1,	0,	....	0,	1,	0,	....	0,
0,	0,	....	0,	0,	0,	....	0.

These two sets of values *may* be reducible into each other. According as they are or are not so reducible, the type has one or two groups. I have not succeeded either in effecting the reduction or in proving its impossibility; nor have I been able to find an invariant by whose different values the groups of the type could be distinguished.

There are, then, *at most* two groups of the type $(1)(11 \ldots . 1)$, $p=2$; they are generated by elements satisfying the following relations:

$$\begin{array}{ll|c|c|l}
a^2 & = & 1 & 1 & \\
b_1^2 & = & a & 1 & \\
b_{\rho+1}^2 & = & a & 1 & \\
b_i^2 & = & 1 & 1 & i=2,\,3,\ldots .\,\rho,\ \rho+2,\ldots .\,2\rho.
\end{array}$$

In both groups

$$\begin{aligned}
ab_i &= b_i a,\ i=1,\,2,\ldots .\,2\rho,\\
b_j b_i &= ab_i b_j,\ \text{if } i=j\pm\rho,\\
&= b_i b_j \text{ in all other cases.}
\end{aligned}$$

61. *Type* $(1)(1)(11 \ldots . 1)$, $s=2\rho$ units in the last parenthesis.

Every group of this type is generated by elements

$$a,\ b,\ c_i,\ (i=1,\,2,\ldots .\,2\rho),$$

satisfying the relations:

$$\begin{aligned}
&a^p = 1,\\
&b^p = a^m,\\
&\left.\begin{aligned} c_i^p &= a^{n_i} b^{f},\\ c_i b &= a^{q_i} b c_i,\end{aligned}\right\} i=1,\,2,\ldots .\,2\rho,\\
&c_j c_i = a^{t_{ij}} b^{h_{ij}} c_i c_j,\ i,j=1,\,2,\ldots .\,2\rho, \text{ independently},\\
&ab = ba,\\
&ac_i = c_i a,\ i=1,\,2,\ldots .\,2\rho.
\end{aligned}$$

If in these relations we consider $a=1$, we have (No. 12) the relations satisfied by the generating elements of a group of type $(1)(11 \ldots . 1)$. But we have just considered this type and have seen that the following restrictions may be imposed on the exponents in the generating relations without thereby excluding any group of the type:

$$\begin{aligned}
m_i &\equiv \quad 0 \pmod p, \text{ unless } i=1 \text{ or } \rho+1,\\
n_{ij} &\equiv \begin{cases} 1 \pmod p, \text{ if } i=i+\rho,\\ -1 \pmod p, \text{ if } j=i-\rho,\\ 0 \pmod p \text{ in all other cases.}\end{cases}
\end{aligned}$$

We may, therefore, without excluding any groups, impose the same restrictions on the corresponding exponents in the present type, viz:

$$f_i \equiv 0 \pmod p, \text{ unless } i = 1 \text{ or } \rho + 1,$$

$$h_{ij} \equiv \begin{cases} 1 \pmod p, \text{ if } j = i + \rho, \\ -1 \pmod p, \text{ if } j = i - \rho, \\ 0 \pmod p \text{ in all other cases.} \end{cases}$$

By the usual methods it is seen that the necessary and sufficient condition that no element other than a and its powers be commutative with every element of the group, is that *at least one of the integers* q_i *be not congruent zero* (*mod* p).

62. Most of the relations resulting from transforming the given relations by each of the generating elements are products and powers of the given relations. In the following case, however (among others), the resulting relations are not so expressible.

Transforming the relation

$$c_j c_i = a^{t_{ij}} b^{h_{ij}} c_i c_j,$$

by c^κ, the relation results $\quad a^{h_{ij} q_\kappa + h_{j\kappa} q_i + h_{\kappa i} q_j} = 1,$

whence (note No. 20),

$$h_{ij} q_\kappa + h_{j\kappa} q_i + h_{\kappa i} q_j \equiv 0 \pmod p, \; i, j, \varkappa = 1, 2, \ldots . \; 2\rho, \text{ independently.}$$

If $\rho = 1$, the indices $i, j, \varkappa$ cannot all be distinct, but at least two must be equal, say $j = \varkappa$, and the congruence above becomes

$$h_{ij} q_j + h_{jj} q_i + h_{ji} q_j \equiv 0 \pmod p.$$

This is an identity, since $h_{ij} \equiv - h_{ji}$, $h_{jj} \equiv 0 \pmod p$.

But if $\rho > 1$, we can choose i arbitrarily, and $j \mp i \pm \rho$, $\varkappa = j \pm \rho$, when the congruence becomes

$$h_{ij} q_{j \pm \rho} + h_{j, j \pm \rho} q_i + h_{j \pm \rho, i} q_j \equiv 0 \pmod p, \; i = 1, 2, \ldots . \; 2\rho.$$

But, since $j \mp i \pm \rho$,

$$h_{ij} \equiv 0, \; h_{j \pm \rho, i} \equiv 0, \; h_{j, j \pm \rho} \not\equiv 0 \pmod p.$$

Substituting these values we have

$$h_{j, j \pm \rho} q_i \equiv 0 \pmod p, \; i = 1, 2, \ldots . \; 2\rho.$$

Whence $\quad q_i \equiv 0 \pmod p, \; i = 1, 2, \ldots . \; 2\rho.$

This is a necessary consequence of the generating relations given. But we have seen that, in order that the elements satisfying the relations given shall generate a group of the type under consideration, it is necessary that at least one of the quantities q_i be *not* congruent zero, (mod p). *There are, therefore, when* $\rho > 1$, *no groups of the type* (1)(1)(11 . . . . 1). The case $\rho = 1$, type (1)(1)(11), has already been considered.

CLARK UNIVERSITY, WORCESTER, MASS., *August* 12, 1892.

The Projection of Fourfold Figures upon a Three-Flat.

By T. Proctor Hall.

1.—*Introductory.*

Stringham has shown* that in space of any dimensions there are at least three regular figures. These three series of figures may be developed synthetically from a point, each according to its own law of increase; and in such ways as this it is possible to develop a synthetic geometry of higher space along with, perhaps, such distinctness of conception as we now have of space of one, two and three dimensions.

Hinton,† indeed, maintains not only that such a conception of four-fold space is possible, but that it can be attained with comparative ease by a careful synthetic study of a few four-fold figures.

I have adopted as most convenient and complete the nomenclature in which the order of an n-fold solid is indicated by a word designating the number of its axes; as tessaract, pentact, etc.; and in which the particular solids of each order are distinguished by numeral adjectives referring to the number of $(n-1)$-fold boundaries.

A point moving in one dimension traces a straight line. Keeping its extremities fixed, suppose the line broadened into an equilateral triangle by extension of its middle point in a direction perpendicular to the line. Again, suppose the middle point of the triangle extended along the third rectangular axis, carrying with it the plane so as to form a regular tetrahedron. Now let the middle point of the tetrahedron be extended in a direction perpendicular to the first three axes, carrying with it the four solid tetrahedra which extend from this central point to each tetrahedral face, until each of these becomes a regular tetrahedron. The enclosed four-fold figure is a penta-tessaract. In these processes one new angular point is added for each new dimension; a new line is

* Am. J. Math., Vol. 3, 1880, p. 6.

† "A New Era of Thought" (Swan, Sonnenschein & Co., London, 1888). For the literature of higher space see Am. J. Math., Vol. 1, 1878, pp. 261, 384 et seq., and Vol. 2, p. 65.

added for each point in the preceding figure; a new plane for each line, and so on; so that the law of increase of boundaries in this triangular series is very simple. See the table of figures following.

SERIES OF REGULAR FIGURES IN n-FOLD SPACE.

Series.	n	FIGURES.	Number of Boundaries.						
			Points.	Lines.	Planes.	Solids.	Tessaracts.	Pentacts.	Hexacts.
Triangular.	1	Line, ...	2						
	2	Triangle, ...	3	3					
	3	Tetrahedron, ...	4	6	4				
	4	Penta-tessaract, ...	5	10	10	5			
	5	Hexa-pentact, ...	6	15	20	15	6		
	6	Hepta-hexact, ...	7	21	35	35	21	7	
	7	Okta-heptact, ...	8	28	56	70	56	28	8
		Etc.							
Rectangular.	1	Line, ...	2						
	2	Square, ...	4	4					
	3	Cube, ...	8	12	6				
	4	Tessaract, ...	16	32	24	8			
	5	Pentact, ...	32	80	80	40	10		
	6	Hexact, ...	64	192	240	160	60	12	
	7	Heptact, ...	128	448	672	560	280	84	14
		Etc.							
Oktahedral.	1	Line, ...	2						
	2	Square, ...	4	4					
	3	Oktahedron, ...	6	12	8				
	4	Hexadeka-tessaract, . .	8	24	32	16			
	5	Triakontaduo-pentact, .	10	40	80	80	32		
	6	Hexakontatetra-hexact,	12	60	160	240	192	64	
	7	128-heptact, ...	14	84	280	560	672	448	128
		Etc.							

Next suppose the line generated by a point to move along the second axis bodily, so as to generate a square. The square moving along the third axis generates a cube; and the cube, moving along the fourth axis, a rectangular tessaract (called for brevity simply a tessaract). In this series each point produces in the new figure a new line and point; each line a new square and line, each square a new cube and square, and so on according to this simple law of increase illustrated in the preceding table.

In picturing the generation of the oktahedral series we must suppose the point extended in two opposite directions to form a line; the middle (original) point of the line extended both ways along the second axis to form a square; the same point extended both ways along the third axis to form a regular oktahedron; along the fourth axis to form a hexadeka-tessaract; and so on. In this series two new angular points are added for each dimension, and each time the number of $(n-1)$-fold boundaries is doubled. The simplest law of the numbers of boundaries is found by observing that each figure of this series is the inverse of the corresponding figure of the rectangular series, and consequently each horizontal row of numbers in the oktahedral series in the preceding table must be the same as the corresponding row in the rectangular series taken in the reverse order. This conclusion may be easily confirmed by working out the slightly complex law of increase in the same way as for the other series.

I have selected the tessaract, the hexadeka-tessaract and the penta-tessaract as simple and typical illustrations of the synthetic method in projection, the simplicity and ease of which becomes more apparent as it is used.

2.—*Projection of a Square upon a Plane.*

The principles of parallel projection may be illustrated by considering briefly this case. Let a and b be the axes of the square, i. e. lines joining the middle points of opposite sides, and let N be a line through the center of the square and normal to it. Let P indicate the line of projection, which I shall take to be always perpendicular to the flat upon which the projection is made. The reader may follow more readily if he marks the axes on a square card, with a pin through its center to represent N.

Case I.—N parallel to P, and therefore a and b perpendicular to P.

The projection is an equal square.

Case II.—a parallel to P.

a becomes a point in the projection; b is unchanged; and the projected figure is therefore a line of unit length.

Case III.—N and a inclined, b perpendicular to P.

a is shortened; b and the angles at its extremities are unchanged. The projection is therefore a rectangle of unit length whose width may vary from zero to unity as limits.

Case IV.—N perpendicular, a and b inclined to P.

The two axes are projected upon one straight line. The figure is therefore a line whose length is between the limits unity and the square root of two.

Case V.—N, a and b inclined to P.

Both axes are shortened and the angle between them altered in the projection. The figure is that of an oblique parallelogram whose limits are three, a square, its diagonal on the one hand, and its side on the other hand.

The principles illustrated in this simple example may be summarized thus:

1. Parallel lines remain parallel.
2. Lines and angles perpendicular to P are unchanged.
3. Lines are shortened and angles altered when not perpendicular to P.
4. Lines parallel to P are reduced to zero, and angles to zero or 180° when in a plane parallel to P.

3.—*Projection of a Cube upon a Three-flat.*

Assuming now that there is a space of four dimensions, the assumption implies that these principles of projection, as well as all geometric relations that are true of three mutually perpendicular axes of three-fold space, are also true for any three of the four axes of four-fold space. Just as in our projection of the square it was necessary that the square should be outside of the plane upon which it was projected, so must our cube be placed outside of three-fold space in the direction of the fourth dimension, in order that it may be similarly projected upon our three-fold space. Let a, b, c be the three axes of this cube, i. e. lines joining the middle points of opposite faces; and let N be a line through the intersection of the axes, perpendicular to all of them. Let P be, as before, the line of projection, which is perpendicular to the three axes of the three-flat upon which the cube is to be projected. Now the essential difference between this

problem and our last lies in this, that instead of N we have now the third axis c, and have a new N whose location may be for the present inconceivable to us, but which bears the same relation to any one of the three known axes of the cube that these bear to each other.

Case I.—N parallel to P, and therefore a, b, c perpendicular to P.

The axes and angles are unaltered. The figure is a cube.

Case II.—a parallel to P, and therefore N, b and c perpendicular to P.

a becomes zero in the projection; b and c and the angles between them are unaltered. The figure is a square. It is evident also that whenever the cube is so situated that P passes through two points in it, or in other words, when P is in the same three-flat with the cube, the projected figure is plane; so that such cases need not be further considered here.

Case III.—N and a inclined, b and c perpendicular to P.

a is shortened; b and c and the angles between them and at their extremities are unaltered. The figure is therefore a right square prism varying between a square and a cube as limits.

Case IV.—N, a and b inclined, c perpendicular to P.

a and b are shortened and their inclination altered; c and the angles at its extremity are unchanged. The figure is therefore a vertical prism whose base is an oblique parallelogram. The sides of the parallelogram vary from zero to unity, and its greater diagonal from unity to the square root of two.

Case V.—N, a, b and c inclined to P.

All the axes are shortened and all the angles altered. The figure is a doubly oblique parallelopipedon whose limits are all of the figures of the four preceding cases.

The projection of any other regular solid upon a three-flat may be found in a similar way, or it may, in many cases, be inferred from the corresponding projection of the cube by suitable modifications of its boundaries.

4.—*Projection of a Tessaract.*

A tessaract is a four-fold figure bounded by eight cubes, one at each end of its four rectangular axes. These cubes are bounded by twenty-four squares, each of which is a double boundary; the squares by thirty-two lines, each of

which is common to three squares; the lines by sixteen points, each of which is the extremity of four lines. It must be noted that the twenty-four squares do not enclose the tessaract, but appear here and there upon its boundaries as edges do upon a cube; for a tessaract cannot be enclosed by planes any more than a cube can be enclosed by lines. In a three-flat projection, however, the tessaractic content disappears and the projection is bounded by planes, just as the plane projection of a solid is bounded by lines. The form of the faces bounding the projection of a tessaract may be found by consideration of the projection of the faces alone, but they are all determined by the positions and lengths of the projected axes of the tessaract. A tessaract is simply a cube extended to unit distance in the fourth dimension—in the direction of N. When any of the three axes of one of the bounding cubes is parallel to P, that cube is projected as a square. With the same nomenclature as before, except that N is now replaced by the fourth axis d; and designating the nearer cubes at the extremities of a, b, c and d by a_1, b_1, c_1, d_1, and the more distant cubes by a_2, b_2, c_2, d_2, we proceed.

Case I.—a parallel to P, therefore b, c and d perpendicular to P.

a is zero in the projection; b, c, d, and the angles between them are unaltered. The figure is a cube; or, more exactly, two coincident cubes, a_1 and a_2, whose faces are also coincident with the six squares which are the projections of the remaining six cubes of the tessaract.

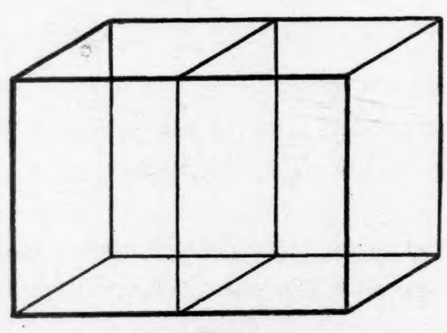

FIG. 1.

Case II.—a and b inclined, c and d perpendicular to P. (Fig. 1.)

a and b fall into a single line; c, d and the angles about them are unaltered. The figure is a right square prism, the length of whose $a+b$ axis is between unity and $\sqrt{2}$, consisting of the projections of a_1 and b_1 side by side, and coinci-

dent with these the pair b_2 and a_2. When a and b are equally inclined, a_2 coincides with b_1, and b_2 with a_1.

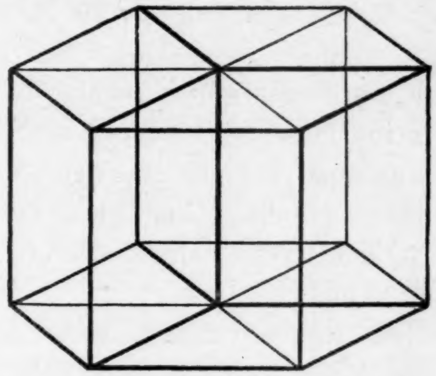

FIG. 2.

Case III.—a, b and c inclined; d perpendicular to P. (Fig. 2.)

a, b and c are shortened and the angles between them altered; d and the angles about it are unchanged. Since d remains perpendicular to all the other axes, a, b and c are in one plane in the projection. The figure is therefore a vertical hexagonal prism, which is regular when a, b and c are equally inclined to P, and in this case consists of a_1, b_1, c_1, coincident with a_2, b_2, c_2 in such a manner that a_2 coincides with the figure formed by the adjacent halves of b_1, c_1; and so on. When the hexagonal prism is not regular it varies toward a cube on one hand and toward the prism of *Case II* on the other hand.

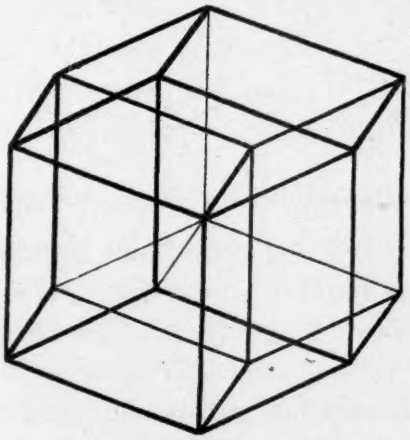

FIG. 3.

Case IV.—All the axes inclined to P. (Fig. 3.)

All the axes are shortened and there are no right angles in the projection. The eight cubes appear as two coincident sets of four. When all the axes are equally inclined to P each makes equal angles with the other three in the projection, that is to say, the common cube-edges, which are parallel to the four axes, diverge toward the four corners of a regular tetrahedron. Each of the four cubes, a_1, b_1, c_1, d_1, is in contact with each of the other three by one of its faces, leaving twelve faces external, bounding the figure, a rhombic dodekahedron. Coincident with this is another dodekahedron formed by the other four projected cubes in such a way that a_2 is coincident with the figure formed of the adjacent thirds of b_1, c_1 and d_1, and so on.

Irregular figures of this case are dodekahedra varying toward any of the figures of the first three cases.

Outline models, of wire, can be readily constructed to show the six projected cubes in *Case III*, and the eight in *Case IV*; though it is not easy to see them all in pictures of the models. (See figures.)

Plane projections of a tessaract are identical with plane projections of three-flat projections of the same. It is evident also that three-flat projections of figures in space of five or more dimensions are identical with three-flat projections of certain four-fold figures; so that no visualized conception of five-fold figures appears possible until four-fold conceptions become familiar to us.

An interesting projection of a tessaract is the well-known figure which corresponds to the perspective view of the interior of a cube as seen through its front face. This consists of a large outline cube having a smaller cube suspended at its center by eight strings connecting the eight pairs of adjacent corners. The analogy between the eight cubes thus represented in perspective and the six faces in the perspective plane projection of a cube is complete.

5.—*Projection of a Hexadeka-tessaract.*

A hexadeka-tessaract is a four-fold figure bounded by sixteen tetrahedra whose faces, as in the boundaries of all four-fold figures, are not free but between two solids. These thirty-two triangular faces are bounded by twenty-four lines, each line being common to four faces. From each of the eight angular points six lines pass to all the other points except the opposite extremity of the same axis. Since these eight points are situated at the ends of four equal and mutually perpendicular axes, the three-flat projections are easily outlined by joining

the extremities of the projected axes as found in the projection of the tessaract. I give here simply the resulting figures under the different cases, and designate by a_1, a_2, the extremities of the axis a, etc.

Case I.—a parallel to P.

a_1 and a_2 are coincident at the intersection of the axes. The figure is a regular oktahedron composed of eight tetrahedra which extend from its center to each of its faces.

Case II.—a and b inclined to P.

An oktahedron whose axes are $1, 1, \frac{1}{2}\sqrt{2}$ is the figure when a and b are equally inclined. In other cases as the short axis approaches unity the points a_1, a_2, which are connected by lines to all the remaining angular points, move symmetrically from the extremities toward the center of the short axis.

Case III.—a, b and c inclined to P.

The typical form is that of two hexagonal prisms, base to base, having its vertical axis longer than the three horizontal axes, and enclosing twelve tetrahedra. This form varies toward that of *Case I* by the contraction of one of the horizontal axes toward its center while the other two extend toward unity in length and approach perpendicularity. It varies toward the regular form of *Case II* when two of the horizontal axes approach coincidence in direction and unity in length. The mode of variation toward the other forms of *Case II* is evident.

Case IV.—All the axes inclined to P.

When the axes are equally inclined the resulting figure is a cube, each of whose faces is crossed by two diagonal lines which complete the outlines of the sixteen tetrahedra included in the cube. The variations from this toward the forms of *Cases I*, *II* and *III* are readily seen.

6.—*Rotation.*

The only rotation possible in a plane is rotation about a point. In threefold space rotation about a point is also rotation about a line. Rotation is essentially motion in a plane, and when another dimension is added to the rotating body, another dimension is added also to the axis of rotation. In fourfold space, accordingly, every rotation takes place about a fixed axial plane. Rotation implies the motion of only two rectangular axes. All other axes perpendicular to these are not affected by it. Of the six mutually perpendicular

planes of a tessaract when rotation takes place in one, one other remains fixed and the other four move in a manner analogous to that of the two which pass through the fixed axis of a rotating cube.

The meaning of rotation about a plane becomes clearer when we consider its projection. In two-fold space the projection upon a line of a line rotating about one end is a line whose other extremity has a simple harmonic motion across the fixed point. In three-fold space the plane-projection of a rectangle rotating about one of its sides is (*a*) a line rotating about one extremity, (*b*) a rectangle, one side of which executes a simple harmonic motion across the opposite side which remains fixed, or (*c*) a parallelogram, one of whose sides moves elliptically about the opposite side. In four-fold space the three-flat projection of a cube rotating about one of its faces is (*a*) a square rotating about one of its sides, (*b*) a cube, one of whose faces executes a simple harmonic motion through the opposite face, or (*c*) a parallelopipedon, one of whose faces moves elliptically about the opposite face, so that twice during each rotation the parallelopipedon may become a plane. Rotation of a tessaract when the axial plane is perpendicular to P takes the form (*b*) in projection, and may be well illustrated in models constructed of wire and having hinge-joints. I have constructed such a model to show the change of Fig. 3 into Fig. 2, and conversely, as the tessaract is rotated. One of the four diagonals in Fig. 3 is made in two parts which telescope. Its upper half carries with it the ends of the three boundary wires attached to its extremity, and also the lower halves of the three diagonals parallel to them, so that as it sinks into the lower half of the diagonal the fourth cube is reduced to a plane. In other words, two coincident middle points in Fig. 3, each carrying three of the half-diagonals, move apart along the fourth diagonal, until they reach the middle points of its halves, where they coincide each with one of the extreme points of the same diagonal which have moved along to meet them.

The only transformations (apparent) in the physical world which correspond to rotation about a plane are, so far as I know, the formation of reflected images. With a thin shell we may rudely imitate the change. A glove rotated about a plane would fit the other hand, and we imitate the change by turning the glove inside out. True plane-rotation would reverse it, leaving it right side out.

7.—*Three-flat Projection of a Tetrahedron.*

Let N be a normal to any three rectangular axes of the tetrahedron; and suppose the tetrahedron so placed in four-fold space that N is parallel to P, the

line of projection. In this position the projection is exactly like its original. As the tetrahedron rotates about the plane of N and any axis a, its projection rotates about a. On the other hand when N and a are in the plane of rotation, the plane of b and c is fixed while any point of a executes a harmonic oscillation through that plane in the projection. It is evident then that any three-flat projection of a three-fold body may be obtained by supposing the body compressed along any desired axis, the limit of compression being such as will produce the reflected image of the original. This principle holds obviously for n-fold space.

An interesting case is the projection of a three-fold body which rotates about the plane containing any axis a, together with n, a line perpendicular to a and making an angle less than 90° with N. When the angle Nn is 45°, each point in the projection describes an ellipse in such a manner that, as the body makes one revolution about the fixed plane, the projection completes one revolution about a, and also, while rotating, has become gradually contracted along a fixed axis perpendicular to the fixed plane until wholly in that plane, and then returned to its original form. When the angle Nn is less than 45° the contraction does not proceed so far. When it is greater than 45°, the reverse figure is formed and the projection is a plane-figure twice during each revolution.

8.—*Projection of a Pentatessaract.*

In describing the triangular series of figures we supposed the pentatessaract derived from the tetrahedron by an extension of its middle point along the fourth axis. Suppose the pentatessaract placed so that P passes through this last point, d_1, and along the fourth axis, d, to the center of the original tetrahedron. Evidently the projection is the original tetrahedron with four others extending from its center to its four faces. Rotation about a plane containing d and any other axis, a, is projected as rotation about a. Rotation about any axial plane not containing d leaves that plane unaltered in the projection; while, as the figure rotates in the plane of d and any other axis, a; d is projected in the same straight line with a; and as d is extended a is shortened in the projection. Any projection of the pentatessaract may therefore be obtained by moving the point d_1 from the center of the tetrahedron along the line of any axis, a, while at the same time the tetrahedron is compressed proportionately along the axis a. The point d_1 of course remains connected with each of the other four angular points, marking out, in general, the other four tetrahedra.

March 18, 1892.

Note on a Geometrical Theorem.

By C. N. Little.

Let *abcdef* be a hexagon having a Pascal line *Lp* and a Brianchon point *Pb*.

From this 6-gon let other 6-gons in infinite succession be derived: (a) by prolonging to intersection alternate sides as *ab*, *cd* and joining adjacent points of intersection; (b) by connecting alternate vertices as *ac*, *bd* and noting their intersection for vertices of the succeeding 6-gon.

Theorem: Every 6-gon so formed will have *Lp* and *Pb* as Pascal line and Brianchon point respectively.

That all must have *Pb* in common follows because the 6-gon next larger than *abcdef* has as cross diagonals Pascals that must pass through *Pb*. In fact they are the three Pascals passing through the Steiner *g* point* $\left\{\begin{matrix} ad, & cb, & ef \\ be, & fa, & dc \\ cf, & de, & ba \end{matrix}\right\}$ which coincides with *Pb*.

But from the theory of reciprocal figures, since all 6-gons have *Pb* in common, and *abcdef* has the Pascal *Lp*, all must have *Lp* in common.

Nebraska State University, Lincoln, *July*, 1892.

* Notation that of Salmon, "Conic Sections," p. 380.

On Groups whose Orders are Products of Three Prime Factors.

By F. N. Cole and J. W. Glover.

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§1.

Introduction.

The general problem of the construction of groups of given orders has as yet received comparatively little attention. This is probably due to the fact that the theory of groups has been developed for the most part under the particular phase of the theory of substitutions, in which the operations of a group appear under the form of permutations of letters, and the groups themselves are primarily classified, not according to their order, but according to their degree, i. e. the number of letters affected. Thus we possess a very elaborate series of general theorems limiting the number and character of groups of given degree, while the construction of groups of given orders has thus far been effected only in the most elementary cases. It is, for example, as yet unknown how many and what groups there are of order 24. The foundation for the treatment of groups of given order is, however, already laid in the theory of the composition of groups which, though usually treated as a part of the theory of substitutions, belongs in reality to the pure theory of groups.*

Important and fertile as the consideration of substitution groups is for the theory of equations and of algebraic relations, it is clear that in treating the structure of a group great advantage is gained by discarding all unessential features and regarding the group from its purely formal side, as a group in the abstract, without reference to the content to which its operations may be applied.

In conformity with this idea, Cayley (American Journal, Vols. I and XI) and Kempe (Phil. Trans., Vol. CLXXVII) have determined the groups according to

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In conformity with this idea, Cayley (American Journal, Vols. I and XI) and Kempe (Phil. Trans., Vol. CLXXVII) have determined the groups according to

* Cf. Sylow's theorems below.

their orders as far as order 12. It is, however, a defect of this method of classification that it proceeds simply according to the order and not the *type* of the groups. Thus, the groups of order pq, where p and q are prime numbers, are all of one of two types, the orders 10, 14, 15, ... presenting no greater complexity than the order 6.

The type of a group depends on the number and mutual relations among the prime factors of its order. Thus the groups of order pq are of only one or of two types according as $p - 1$ is not or is divisible by q. The difficulty of construction naturally increases very rapidly with the number of factors. But if this number does not exceed three, the difficulty is not serious, owing to the easily demonstrated presence of a self-conjugate subgroup in every case.

The groups of prime order are, of course, cyclical. Those of order pq and p^2 are also known.* In the following, those whose order is a product of three, equal or unequal, prime factors are fully determined. These three classes alone comprise 371 of the first 500 orders.

§2.

Sylow's Theorems: The Groups of Order p^2 and pq.

Throughout the following we make constant use of two fundamental theorems of Sylow† which are quite indispensable in the dissection of groups:

I. *A group of order p^a, where p is a prime number, always contains a self-conjugate subgroup of order p^{a-1}.*

II. *If the order u of a group is divisible by p^a, but by no higher power of the prime number p, the group contains subgroups of order p^a. These subgroups are all conjugate, and their number is $xp + 1$ where*

$$u = p^a i\,(xp + 1),$$

i being an integer.

By way of illustration we apply these theorems to the analysis of the groups of order p^2 and pq, where p and q are prime numbers.

I. A group of order p^2 may contain an operation of order p^2. The group is then cyclical, and requires no further consideration. Otherwise its actual operations are all of order p, and the powers of one of these must form a self-conjugate subgroup (Theorem I). Suppose this to be $\{s\} = (1,\ s,\ s^2,\ \ldots .\ s^{p-1})$. Any

* Cf. Netto, Theory of Substitutions, pp. 146–9.

† Cf. L. Sylow, Math. Ann. V, pp. 584–94.

other operation t of the group must transform s into one of its powers; for $\{s\}$ is self-conjugate. Suppose that

$$t^{-1}st = s^{\mu}.$$

Then we have, on repeated transformation by t,

$$t^{-2}st^2 = s^{\mu^2},\ t^{-3}st^3 = s^{\mu^3}, \ldots . t^{-p}st^p = s^{\mu^p}.$$

But, as $t^p = 1$, it follows that $t^{-p}st^p = s$. Hence

$$\mu^p \equiv 1 \pmod p.$$

This is only possible if $\mu = 1$. Consequently

$$t^{-1}st = s \text{ or } ts = st.$$

The operations s and t are therefore commutative, and accordingly all their combinations can be reduced to the form $s^{\alpha}t^{\beta}$. There are precisely p^2 operations of this form, and these make up the required group. The latter may be written

$$\begin{array}{lllll}
1 & , s & , s^2 & , \ldots . s^{p-1} & , \\
t & , st & , s^2t & , \ldots . s^{p-1}t & , \\
t^2 & , st^2 & , s^2t^2 & , \ldots . s^{p-1}t^2 & , \\
\cdots & \cdots & \cdots & \cdots & \\
t^{p-1}, & st^{p-1}, & s^2t^{p-1}, & \ldots . s^{p-1}t^{p-1}. &
\end{array}$$

II. Again, a group of order pq may contain an operation of order pq, and is then a cyclical group; otherwise its actual operations are all of orders p or q. By theorem II, there is always one and only one subgroup of order p, which is therefore self-conjugate. Suppose this to be $\{s\} = (1, s, s^2, \ldots . s^{p-1})$. Some of the remaining operations are of order q, and any one of these, t, must transform s into one of its powers. Suppose that

$$t^{-1}st = s^{\mu}.$$

Then, as before, we obtain

$$t^{-q}st^q = s^{\mu^q},$$

where, since $t^q = 1$,

$$\mu^q \equiv 1 \pmod p.$$

If $p - 1$ is not divisible by q, this congruence has only one root, viz. 1. In this case s and t are commutative, and consequently st is of order pq. The group is then cyclical.

But if q is a divisor of $p - 1$, the above congruence admits of $q - 1$ roots different from 1. If any one of these is μ_1, they may be written

$$\mu_1, \mu_1^2, \mu_1^3, \ldots . \mu_1^{q-1} \pmod p.$$

Whichever one of these $q-1$ values of μ is chosen, the resulting group will be the same. For if t transforms s into s^{μ_1}, the successive powers of t will transform s into $s^{\mu_1^2}, s^{\mu_1^3}, \ldots.$ And as all the powers of t, except the q^{th}, are of order q, and as every one of them is a power of every other, it is indifferent which one of them is originally taken for t.

Beside the p operations $\{s\}$ the group contains $pq-p$ of order q. These must distribute themselves in p conjugate groups of order q. These groups are transformed in a cycle by the powers of s. In fact, from

$$t^{-a}st^{a}=s^{\mu_1^a}$$

follows

$$s^{-1}t^{a}s=t^{a}s^{1-\mu_1},\ s^{-2}t^{a}s^{2}=t^{2a}s^{2(1-\mu_1^a)},\ s^{-3}t^{a}s^{3}=t^{3a}s^{3(1-\mu_1^a)},\ldots.,$$

where every system $s^{-i}t^{a}s^{i}$ $(\alpha=0\ldots.q-1)$ evidently furnishes a distinct group.

This second type of group of order pq may therefore be written

$$\begin{array}{lllll}
1 & ,\ s & ,\ s^2 & ,\ \ldots.\ s^{p-1} & , \\
t & ,\ ts^{1-\mu_1} & ,\ ts^{2(1-\mu_1)} & ,\ \ldots.\ ts^{(p-1)(1-\mu_1)} & , \\
t^2 & ,\ t^2s^{1-\mu_1^2} & ,\ t^2s^{2(1-\mu_1^2)} & ,\ \ldots.\ t^2s^{(p-1)(1-\mu_1^2)} & , \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \\
t^{q-1} & ,\ t^{q-1}s^{1-\mu_1^{q-1}} & ,\ t^{q-1}s^{2(1-\mu_1^{q-1})} & ,\ \ldots.\ t^{q-1}s^{(p-1)(1-\mu_1^{q-1})} & ,
\end{array}$$

or, in a form in which the conjugate relation is accentuated,

$$\begin{array}{lllll}
1 & ,\ s & ,\ s^2 & ,\ \ldots.\ s^{p-1} & , \\
t & ,\ s^{-1}ts & ,\ s^{-2}ts^2 & ,\ \ldots.\ s^{-(p-1)}ts^{p-1} & , \\
t^2 & ,\ s^{-1}t^2s & ,\ s^{-2}t^2s^2 & ,\ \ldots.\ s^{-(p-1)}t^2s^{p-1} & , \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \\
t^{q-1} & ,\ s^{-1}t^{q-1}s & ,\ s^{-2}t^{q-1}s^2 & ,\ \ldots.\ s^{-(p-1)}t^{q-1}s^{p-1} & .
\end{array}$$

The same results can of course be obtained by the aid of the theory of substitutions,* but the employment of unnecessary apparatus naturally obscures the real process. Moreover, when any group of operations is known, an isomorphic substitution group can be written down at once.† For this purpose we suppose the operations of the group to be numbered in any order. If now we multiply them all by any one among them, s, the resulting products are the same operations again, but in a different order. Corresponding to s, we have therefore a

*Cf. Netto: loc. cit.

† Cayley, Amer. Jour., Vol. I, p. 51.

permutation of the numbers designating the several operations. And this permutation evidently affects every number. A similar result holds for any other operation t, and so on. Then, since the original operations form a group, the same is true of the permutations. And these two groups are simply isomorphic. The permutation group, moreover, since all its operations except identity affect every number, is a regular group,* and its degree and order are equal.

There is another method by which, given a group of operations, an isomorphic (transitive) substitution group can frequently be found with its degree lower than its order. For example, the not cyclical group of order pq contains p conjugate subgroups of order q. If these are transformed with respect to all the operations in the group, they will simply be permuted among themselves, and these permutations form a group isomorphic with the given one. Moreover, as there is no operation except identity in the group of order pq which, employed as a transformer, leaves all the subgroups of order q unchanged, the group of permutations will also contain pq operations, and the isomorphism is in this case holoedric.

Thus, if in the not cyclical group of order 6,

$$\begin{array}{lll} 1, & s, & s^2, \\ t, & s^{-1}ts, & s^{-2}ts^2, \\ t^2, & s^{-1}t^2s, & s^{-2}t^2s^2, \end{array}$$

the subgroups of order 3, $\{t\}$, $\{s^{-1}ts\}$, $\{s^{-2}ts^2\}$ are numbered 1, 2 and 3, we readily find for the permutations corresponding to s and t,

$$s = (1, 2, 3), \; t = (2, 3),$$

and these generate the required substitution group, which is, of course, the symmetric group of three elements.

In regard to the types of order p^2 and pq, we note further that these can be readily modified so as to furnish groups of higher types. Thus the characteristic feature of the not cyclical group of order pq is that it contains one subgroup of one order and a system of conjugate subgroups of another order, the latter being joined by transformation with respect to the operations of the former. It will at once suggest itself that we may, by way of trial, replace the subgroup of

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* Cf. Bolza, Amer. Jour., Vol. XIII, p. 23. It is easily seen that the permutation group above is *transitive*.

order p by a group of a higher type, for example, p^2, at the same time properly increasing the number of subgroups of order q, or replacing these also by a proper number of groups of a higher type. (Cf. the groups α_2, α_3, β_2, γ_3, γ_4 below.)

§3.

The Groups of Order p^3.

From the first of Sylow's theorems it appears that a group of order p^3 always contains a self-conjugate subgroup of order p^2. We show further that

If a group of order p^3 contains a cyclical subgroup of order p^2, it contains a self-conjugate cyclical subgroup of order p^2.

For suppose that a G_{p^3} contains an operation s of order p^2. If t is any other operation of G_{p^3}, then t^p is contained among the powers of s^p. Otherwise the products $t^\alpha s^\beta$ would furnish at least p^4 different operations.

If now the group $\{s\}$ is not itself self-conjugate, there will be some operation t in the G_{p^3} which transforms $\{s\}$ into p conjugate groups. These groups must have the powers of s^p in common. Beside these they contain $p(p^2-p)$ different operations. There remain, including the powers of s^p, p^2 operations, and as there must always be a self-conjugate subgroup of order p^2, these p^2 operations must constitute this subgroup.

If the latter is cyclical, the theorem is proved. If it is not cyclical, t, which is contained in it, must be of order p and commutative with s^p. The group $\{s^p, t\}$ being self-conjugate, we must then have

$$s^{-1}ts = s^{\alpha p}t^\beta,$$

and $\beta^p \equiv 1 \pmod p$, that is, $\beta = 1$. Then

$$s^{-1}ts = s^{\alpha p}t, \quad \therefore\ tst^{-1} = s^{\alpha p+1},$$

so that t would transform the group $\{s\}$ into itself, which is contrary to assumption.

We may now conveniently divide the groups of order p^3 into two classes, according as they do or do not contain an operation of order p^2.

The former class includes the cyclical group, which, of course, exists for every p. This group, which we designate by α_1, is generated by a single opera-

tion s of order p^3. It contains one subgroup of order p, $\{s^{p^2}\}$, and one of order p^2, $\{s^p\}$, and they are both cyclical and both self-conjugate. There are $p-1$ operations of order p, $p(p-1)$ of order p^2, and $p^2(p-1)$ of order p^3.

We proceed now to discuss the not cyclical groups of order p^3.

A.—*An operation of order p^2 present.*

We have shown that there must in this case be a self-conjugate cyclical subgroup of order p^2. Let this be

$$H_{p^2} \equiv \{s\} \equiv (1, s, s^2, \ldots . s^{p^2-1}).$$

If t is any other operation of the group, we have

$$t^{-1}st = s^{\mu},$$

and $\mu^p \equiv 1 \pmod{p}$. Hence $\mu = \varkappa p + 1$, and

$$t^{-1}st = s^{\varkappa p + 1}.$$

If $\varkappa \neq 0$, we have further

$$t^{-2}st^2 = s^{2\varkappa p+1},\ t^{-3}st^3 = s^{3\varkappa p+1}, \ldots .$$

Accordingly, by replacing t by a proper power of t, we can always arrange that $t^{-1}st = s^{p+1}$. There are therefore two distinct types according as

$$t^{-1}st = s \text{ or } t^{-1}st = s^{p+1}.$$

We show further that in each of these types operations of order p occur which are not contained among the powers of s^p. For the identity $t^{-1}st = s^{\varkappa p+1}$ furnishes

$$(t^{\lambda}s^{\nu})^{\alpha} = t^{\alpha\lambda}s^{\nu\left(\alpha + \frac{\alpha(\alpha-1)}{2}\lambda\varkappa p\right)}.$$

If then $\varkappa = 0$, or if $p > 2$, we have

$$(t^{\lambda}s^{\nu})^p = t^{p\lambda}s^{p\nu},$$

and since $t^{\lambda p} = s^{\omega p}$, the operation $t^{\lambda}s^{-\omega}$ will be of order p. The case $p=2$, $\varkappa \neq 0$ is specially considered at a later point.

We may now assume that t is of order p. The equation above then becomes

$$(t^{\lambda}s^{\nu})^p = s^{\nu p\left(1 + \frac{p(p-1)}{2}\lambda\varkappa\right)},$$

from which it appears that, since $1 + \dfrac{p(p-1)}{2}\lambda\varkappa$ is not divisible by p, all the

operations of order p in the group are contained in the not cyclical subgroup $\{t, s^p\}$. The group G_{p^3} therefore contains $p^3 - p^2$ operations of order p^2. These must be distributed in p cyclical groups of order p^2, having the subgroup $\{s^p\}$ in common. Since the group $\{s\}$ is self-conjugate, the remaining $p - 1$ groups are self-conjugate. Their operations of order p^2, together with the p^2 operations $\{t, s^p\}$, make up the entire group.

We now consider the different types separately.

a). The identity is $t^{-1}st = s$, $(t^p = 1)$.

This group a_2 has all its operations commutative, and its subgroups therefore all self-conjugate. The group always exists. It contains

$p + 1$ self-conjugate subgroups of order p, $\{s^p\}$, $\{ts^{ap}\}$, $(a = 0, 1, \ldots . p - 1)$;
1 self-conjugate, not cyclical subgroup of order p^2, $\{s^p, t\}$;
p self-conjugate cyclical subgroups of order p^2, $\{st^a\}$, $(a = 0, 1, \ldots . p - 1)$.

Its operations are distributed according to their orders as follows:

Order......	1	p	p^2
Number....	1	$p^2 - 1$	$p^2(p - 1)$

To obtain a corresponding substitution group of order 27 we may take

$s = (1, 2, 3, 4, 5, 6, 7, 8, 9)(10, 11, 12, 13, 14, 15, 16, 17, 18)$
$(19, 20, 21, 22, 23, 24, 25, 26, 27)$,
$t = (1, 10, 19)(2, 11, 20)(3, 12, 21)(4, 13, 22)(5, 14, 23)(6, 15, 24)$
$(7, 16, 25)(8, 17, 26)(9, 18, 27)$.

b). The identity is $t^{-1}st = s^{p+1}$, $(t^p = 1)$.

This type evidently differs from the preceding in that the operation s transforms the p subgroups of order p,

$$\{ts^{ap}\}, \ (a = 0, 1, \ldots . p - 1),$$

in a cycle. It has therefore

1 self-conjugate subgroup of order p, $\{s^p\}$, and
p conjugate subgroups of order p, $\{ts^{ap}\}$, $(a = 0, 1, \ldots . p - 1)$.

Otherwise its subgroups and operations are the same as those of the preceding case.

This group a_3 always exists except when $p = 2$. A substitution group of this type of order 27 is generated by

$$s = (1, 2, 3, 4, 5, 6, 7, 8, 9)(10, 11, 12, 13, 14, 15, 16, 17, 18)(19, 20, 21, 22, 23, 24, 25, 26, 27),$$

$$t = (1, 10, 19)(2, 14, 26)(3, 18, 24)(4, 13, 22)(5, 17, 20)(6, 12, 27)(7, 16, 25)(8, 11, 23)(9, 15, 21).$$

c). The case $p = 2$, $\varkappa = 1$.

The corresponding group of order 8 may be written

$$\begin{array}{llll} 1, & s, & s^2, & s^3, \\ t, & ts, & ts^2, & ts^3. \end{array}$$

The general identity

$$(t^\lambda s^\nu)^p = t^{\lambda p} s^{\nu\left(p + \frac{p(p-1)}{2}\lambda\varkappa p\right)}$$

becomes here

$$(ts^\nu)^2 = t^2 s^{4\nu} = t^2.$$

The operations

$$ts^\nu \quad (\nu = 0, 1, 2, 3)$$

are therefore all of order 2 or all of order 4, according as t is of order 2 or 4.

In the former case

$$t^{-1}st = s^3, \quad (t^2 = 1).$$

There are here five subgroups of order 2,

$$\{s^2\},\ \{t\},\ \{ts\},\ \{ts^2\},\ \{ts^3\},$$

of which the first is self-conjugate, while the first and third and again the second and fourth are conjugate with respect to s. The operation s^2 combined with these subgroups gives rise to two not cyclical groups of order 4, which are self-conjugate. There is also the single cyclical subgroup of order 4, $\{s\}$.

This group is simply the "dihedron" group of order 8. (Cf. Klein, Ikosaeder, Chap. I.) A corresponding substitution group is generated by

$$s = (1, 2, 3, 4)(5, 6, 7, 8),$$
$$t = (1, 5)(2, 8)(3, 7)(4, 6).$$

In the second case

$$t^{-1}st = s^3, \quad (t^2 = s^2).$$

Here all the operations are of order 4 except s^2 and 1. There are three self-conjugate cyclical subgroups of order 4, $\{s\}$, $\{ts\}$, $\{t\}$, and one self-conjugate subgroup of order 2, $\{s^2\}$. A corresponding substitution group is generated by

$$s = (1, 2, 3, 4)(5, 6, 7, 8),$$
$$t = (1, 5, 3, 7)(2, 8, 4, 6).$$

B.—*No operation of order p^3 present.*

There must be a self-conjugate not cyclical subgroup of order p^2 present,

$$H_{p^2} \equiv \{s, t\}, \quad (st = ts).$$

Since this group contains $p + 1$ subgroups of order p, any operation u not contained in H_{p^2} must transform one of these groups into itself. If this latter group is $\{s\}$, then

$$u^{-1}su = s.$$

Two cases will then arise according as

$$u^{-1}tu = t, \text{ or } u^{-1}tu = s^\lambda t^\mu.$$

a). *The identities are $t^{-1}st = s$, $u^{-1}su = s$, $u^{-1}tu = t$.*

The operations of this group being all commutative, the combination of any two subgroups of order p gives a not cyclical subgroup of order p^2.

The number of subgroups of order p is $\frac{p^3 - 1}{p - 1} = p^2 + p + 1$. These furnish $\frac{(p^2 + p + 1)(p^2 + p)}{2}$ pairs. But as the $p + 1$ subgroups of order p in a group of order p^2 furnish $\frac{(p + 1)p}{2}$ pairs, there are in all

$$\frac{(p^2 + p + 1)(p^2 + p)}{2} \div \frac{(p + 1)p}{2} = p^2 + p + 1$$

not cyclical subgroups of order p^2. All the subgroups of this group are obviously self-conjugate.

This group, which always exists, we denote by α_4. A corresponding substitution group of order 8 is generated by

$$s = (1, 2)(3, 4)(5, 6)(7, 8),$$
$$t = (1, 3)(2, 4)(5, 7)(6, 8),$$
$$u = (1, 5)(2, 6)(3, 7)(4, 8).$$

b). The identities are $t^{-1}st = s,\ u^{-1}su = s,\ u^{-1}tu = s^{\lambda}t^{\mu}$.

We may take s^{λ} as a generator in place of s. Then

$$u^{-1}tu = st^{\mu},$$

where $\mu^{p} \equiv 1 \pmod{p}$. Hence $\mu = 1$, and

$$u^{-1}tu = st.$$

The operation s, being commutative with both t and u, is commutative with all the operations of the group, and in combination with the latter generates $p+1$ not cyclical subgroups of order p^2. These are all the subgroups of this order, for if there were any other, its operations would be commutative with each other and with s, and we should have the preceding type α_4.

Since $\{s, t\}$ and $\{s, u\}$ are self-conjugate, all the subgroups of order p^2 are self-conjugate. Of the subgroups of order p, $\{s\}$ is self-conjugate and the remaining $p^2 + p$ divide into $p+1$ sets of p conjugate groups each, the p of each set belonging in the same subgroup of order p^2.

This group α_5 always exists if $p > 2$. If $p = 2$ it does not exist, since from

$$u^{-1}tu = st$$

follows in this case

$$tu = ust,\quad (tu)^2 = tutu = tu\,.\,ust = s,$$

so that tu would be of order $p^2 = 4$. That this does not happen for $p > 2$ appears from the formula

$$(u^{\lambda}t^{\mu}s^{\nu})^{\kappa} = u^{\kappa\lambda}t^{\kappa\mu}s^{\kappa\nu + \lambda\mu\frac{\kappa(\kappa-1)}{2}}.$$

A substitution group of order 27 of this type is generated by

$s = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18)$
$(19, 20, 21)(22, 23, 24)(25, 26, 27)$,

$t = (1, 4, 7)(2, 5, 8)(3, 6, 9)(10, 13, 16)(11, 14, 17)(12, 15, 18)$
$(19, 22, 25)(20, 23, 26)(21, 24, 27)$,

$u = (1, 10, 19)(2, 11, 20)(3, 12, 21)(4, 14, 24)(5, 15, 22)(6, 13, 23)$
$(7, 18, 26)(8, 16, 27)(9, 17, 25)$.

Reviewing the preceding results, we observe that there are always five types of groups of order p^3; that these types are invariable if $p > 2$, but that for $p = 2$ the regular types α_3 and α_5 are missing, while two new exceptional types occur. Also it appears that

*In a group of order p^3 every subgroup of order p^2 is self-conjugate.**

* For the general theorem of which this is a particular case cf. Frobenius, Crelle, 101, p. 285.

§4.

The Groups of Order p^2q, $(p>q)$.

From Sylow's theorem follows at once

Every G_{p^2q} *contains a single and therefore self-conjugate subgroup of order* p^2, *and* $1, p$, or p^2 *subgroups of order* q.

There are therefore six principal cases to be considered, since the subgroup of order p^2 may be cyclical or not cyclical.

A.—*The* G_{p^2} *cyclical*; $G_{p^2}=\{s\}=(1, s, s^2, \ldots . s^{p^2-1})$.

a). *A single and therefore self-conjugate subgroup* $H_q=\{t\}=(1, t, t^2, \ldots . t^{q-1})$.

Here
$$s^{-1}ts=t^{\mu},$$
and we must take $\mu=1$. The operations s and t are commutative, and consequently $\sigma=ts$ is of order p^2q. The G_{p^2q} is therefore cyclical. It contains 1 subgroup of order p^2, 1 of order p, 1 of order q, and 1 of order pq, and these are all cyclical and self-conjugate.

The operations of the group are distributed according to their orders as follows:

Order......	1	q	p	pq	p^2	p^2q
Number....	1	$q-1$	$p-1$	$(p-1)(q-1)$	$p(p-1)$	$p(p-1)(q-1)$

This group β_1 of course always exists.

b). p *conjugate subgroups of order* q; $\dfrac{p-1}{q}$ *an integer.*

Let one of the subgroups of order q be
$$H_q=\{t\}=(1, t, t^2, \ldots . t^{q-1}).$$
Then
$$t^{-1}st=s^{\mu}, \quad \therefore s^{-1}ts=ts^{1-\mu}, \quad (\mu^q\equiv 1 \bmod p^2, \ \mu \not\equiv 1).$$
It follows then from
$$t^{-1}s^pt=s^{p\mu}$$
that t and s^p are not commutative. Consequently the powers of s transform $\{t\}$ into p^2 conjugate groups.

The case b), therefore, does not exist.

c). p^2 conjugate subgroups of order q; $\frac{p^2-1}{q}$ an integer.

The p^2 subgroups of order q arise, as we have just seen, by transformation of $\{t\}$ with respect to s. They are

$$\{t\},\ \{ts^{1-\mu}\},\ \{ts^{2(1-\mu)}\}, \ldots . \{ts^{(p^2-1)(1-\mu)}\},$$

or, apart from the order of succession,

$$\{t\},\ \{ts\},\ \{ts^2\}, \ldots . \{ts^{p^2-1}\}.$$

These groups contain $p^2(q-1)$ operations, which with the p^2 operations $\{s\}$ make up the entire group. As there are no operations of order pq, all the subgroups of this order are of the not cyclical type. They are formed in each case by the combination of the self-conjugate subgroup of order p, $\{s^p\}$ with one of the subgroups of order q. They are

$$\{s^p, ts^\nu\},\ (\nu = 0, 1, 2, \ldots . p-1).$$

The subgroups of order q in each of these groups are conjugately connected with respect to the powers of s^p, and the groups themselves are conjugately connected by the remaining powers of s.

This group β_2 is analogous to the not cyclical type of order pq, the group of order p in the latter being here replaced by the cyclical group of order p^2.

The group β_2 exists only if $p-1$ is divisible by q.* Its identity is

$$t^{-1}st = s^\mu,\ (\mu^q \equiv 1 \bmod p^2, \mu \not\equiv 1).$$

It contains

p^2 conjugate subgroups of order q, $\{ts^\nu\}$, $(\nu = 0, 1, \ldots . p^2-1)$;
1 self-conjugate subgroup of order p, $\{s^p\}$;
p conjugate not cyclical subgroups of order pq, $\{s^p, ts^\nu\}$, $\nu = 0, 1, \ldots . p-1)$;
1 self-conjugate cyclical subgroup of order p^2, $\{s\}$.

Its operations are distributed according to their orders as follows:

Order....	1	q	p	p^2
Number..	1	$p^2(q-1)$	$p-1$	$p(p-1)$

* Since there are not cyclical subgroups of order pq.

A substitution group of order 18 of this type is generated by

$$s=(1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9)(10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18),$$
$$t=(1,\ 10)(2,\ 18)(3,\ 17)(4,\ 16)(5,\ 15)(6,\ 14)(7,\ 13)(8,\ 12)(9,\ 11).$$

B.—*The* G_{p^2} *not cyclical*; $G_{p^2}=\{s,\ t\}$, $(st=ts)$.

a). *A single and therefore self-conjugate subgroup* $H_q\equiv\{u\}\equiv(1,\ u,\ u^2,\ldots u^{q-1})$.

In this case we have

$$s^{-1}us=u,\ \ t^{-1}ut=u.$$

The operations of the group are, therefore, all commutative. Each of the $p+1$ groups of order p,

$$\{s\},\ \{t\},\ \{ts\},\ \{ts^2\},\ldots.\{ts^{p-1}\},$$

combines with the subgroup $\{u\}$ to form a cyclical subgroup of order pq. These $p+1$ groups, having the group $\{u\}$ in common, contain beside this

$$(p+1)(pq-q)=(p^2-1)q$$

different operations, which with the powers of u make up the entire group.

This group β_3 always exists. Its identities are

$$s^{-1}ts=t,\ \ s^{-1}us=u,\ \ t^{-1}ut=u,\ \ (s^p=t^p=u^q=1).$$

It contains

1 self-conjugate subgroup of order q, $\{u\}$;

$p+1$ self-conjugate subgroups of order p, $\{s\}$, $\{ts^\nu\}$, $(\nu=0,\ 1,\ldots.p-1)$;

$p+1$ self-conjugate cyclical subgroups of order pq,

$$\{u,\ s\},\ \{u,\ ts^\nu\},\ (\nu=0,\ 1,\ldots.p-1);$$

1 self-conjugate not cyclical subgroup of order p^2, $\{s,\ t\}$.

Its operations are distributed according to their orders as follows:

Order....	1	q	p	pq
Number.	1	$q-1$	p^2-1	$(p^2-1)(q-1)$

A substitution group of order 18 of this type is generated by

$$s=(1,\ 2,\ 3)(4,\ 5,\ 6)(7,\ 8,\ 9)(10,\ 11,\ 12)(13,\ 14,\ 15)(16,\ 17,\ 18),$$
$$t=(1,\ 4,\ 7)(2,\ 5,\ 8)(3,\ 6,\ 9)(10,\ 13,\ 16)(11,\ 14,\ 17)(12,\ 15,\ 18),$$
$$u=(1,\ 10)(2,\ 11)(3,\ 12)(4,\ 13)(5,\ 14)(6,\ 15)(7,\ 16)(8,\ 17)(9,\ 18).$$

b). *p conjugate subgroups of order q*; $\frac{p-1}{q}$ *an integer.*

Since q is a divisor of $p-1$, any operation u of order q must transform at least two of the $p+1$ subgroups of order p into themselves.* Suppose these to be $\{s\}$ and $\{t\}$. Also either s or t, suppose s, must transform $\{u\}$ into itself; otherwise there would be more than p subgroups of order q. We have therefore the following identities

$$t^{-1}st=s,\ u^{-1}su=s,\ u^{-1}tu=t^{\mu},$$

where $\mu \neq 1$, since $\mu=1$ leads to the preceding case.

The operations t and u generate a not cyclical subgroup of order pq. This subgroup is evidently self-conjugate, since s is commutative with both t and u. It contains all the operations of order q. The latter being all commutative with s, there are p conjugate cyclical subgroups of order pq. These have the group $\{s\}$ in common, and contain beside this $p(pq-p)$ different operations, which with the p^2 operations $\{s, t\}$ make up the entire group.

The p subgroups of order q are conjugately connected by the operation t. Two of the subgroups of order p, viz. $\{s\}$ and $\{t\}$ are self-conjugate. The remaining $p-1$ subgroups of this order divide into $\frac{p-1}{q}$ sets of q groups conjugately connected by u.

This group β_4 exists only when $p-1$ is divisible by q. Its identities are

$$t^{-1}st=s,\ u^{-1}su=s,\ u^{-1}tu=t^{\mu}\quad (\mu \neq 1),\ (s^p=t^p=u^q=1).$$

It contains

p conjugate subgroups of order q, $\{t^{-a}ut^a\}$, $(a=0, 1, \ldots. p-1)$;

2 self-conjugate subgroups of order p, $\{s\}$, $\{t\}$;

$\frac{p-1}{q}$ sets of q conjugate subgroups of order p, $\{ts^a\}$, $(a=1, 2, \ldots. p-1)$;

1 self-conjugate not cyclical subgroup of order pq, $\{u, t\}$;

p conjugate cyclical subgroups of order pq, $\{t^{-a}ut^a, s\}$, $(a=0, 1, \ldots. p-1)$;

1 self-conjugate not cyclical subgroup of order p^2, $\{s, t\}$.

Its operations are distributed according to their orders as follows:

Order.....	1	q	p	pq
Number...	1	$p(q-1)$	p^2-1	$p(p-1)(q-1)$

***The case $p=2$ presents no exception. In this case u might *interchange* $\{t\}$ and $\{s\}$. But then u would leave $\{st\}$ and consequently some other $\{s^rt\}$ unchanged.**

A substitution group of order 18 of this type is generated by

$$s = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18),$$
$$t = (1, 4, 7)(2, 5, 8)(3, 6, 9)(10, 13, 16)(11, 14, 17)(12, 15, 18),$$
$$u = (1, 10)(2, 11)(3, 12)(4, 16)(5, 17)(6, 18)(7, 13)(8, 14)(9, 15).$$

c). p^2 conjugate subgroups of order q; $\frac{p^2-1}{q}$ an integer.

α). $\frac{p-1}{q}$ an integer.

As in the preceding case, every operation u of order q must transform two of the $p+1$ subgroups of order p into themselves. If these are taken for $\{s\}$ and $\{t\}$, we have

$$u^{-1}su = s^{\mu}, \quad u^{-1}tu = t^{\nu}.$$

Here μ and ν must both be different from 1; otherwise we should have either β_3 or β_4. The p^2 subgroups of order q are obtained by transforming $\{u\}$ with respect to the p^2 operations $\{s, t\}$.

We have now to distinguish two cases according as μ and ν are equal or unequal.

If $\mu = \nu$, every operation of order q transforms every operation $s^{\alpha}t^{\beta}$ of order p into its μ^{th} power. Accordingly, any operation of order p combined with any operation of order q generates a not cyclical subgroup of order pq. There are p of these groups containing s, p containing t, and so on; in all $p(p+1)$. Since $\{s\}$, $\{t\}$, ... are each self-conjugate, these $p(p+1)$ groups divide into $p+1$ sets of p conjugate groups each.

This group β_5 exists only if $p-1$ is divisible by q. It is analogous to the not cyclical type of order pq, the subgroup of order p in the latter being here replaced by the not cyclical subgroup of order p^2. The identities are

$$t^{-1}st = s, \quad u^{-1}su = s^{\mu}, \quad u^{-1}tu = t^{\mu}, \quad (\mu^q \equiv 1 \bmod p).$$

The group contains

p^2 conjugate subgroups of order q, $\{(t^{\alpha}s^{\beta})^{-1}u(t^{\alpha}s^{\beta})\}$, $(\alpha, \beta = 0, 1, \ldots. p-1)$;

$p+1$ self-conjugate subgroups of order p, $\{s\}$, $\{ts^{\nu}\}$, $(\nu = 0, 1, \ldots. p-1)$;

$p+1$ sets of p conjugate not cyclical subgroups of order pq,

$\{s, u\}$, $\{ts^{\nu}, u\}$, $(\nu = 0, 1, \ldots. p-1)$

1 self-conjugate not cyclical subgroup of order p^2, $\{s, t\}$.

The operations are distributed according to their orders as follows:

Order	1	q	p
Number ..	1	$p^2(q-1)$	p^2-1

A substitution group of order 18 of this type is generated by

$s=(1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18)$,

$t=(1, 4, 7)(2, 5, 8)(3, 6, 9)(10, 13, 16)(11, 14, 17)(12, 15, 18)$,

$u=(1, 10)(2, 12)(3, 11)(4, 16)(5, 18)(6, 17)(7, 13)(8, 15)(9, 14)$.

If $\mu \neq \nu$, only $\{s\}$ and $\{t\}$ can be combined with operations u to generate not cyclical subgroups of order pq. Instead of $p(p+1)$ such groups we have here only $2p$, which divide into two sets of p conjugate groups each.

This group β_6 exists only if $p-1$ is divisible by q. The case $q=2$ is inadmissible under this type, since the congruence $\mu^2\equiv 1 \pmod p$ has only one root different from 1. The identities are

$$t^{-1}st=s,\ u^{-1}su=s^{\mu},\ u^{-1}tu=t^{\nu},\ (\mu^q\equiv 1,\ \nu^q\equiv 1 \bmod p,\ \mu\neq\nu).$$

The group contains

p^2 conjugate subgroups of order q, $\{(s^{\alpha}t^{\beta})^{-1}u(s^{\alpha}t^{\beta})\}$, $(\alpha, \beta=0, 1, \ldots . p-1)$;

2 self-conjugate subgroups of order p, $\{s\}$, $\{t\}$;

$\dfrac{p-1}{q}$ sets of q conjugate subgroups of order p, $\{ts^{\nu}\}$ $(\nu=1, 2, \ldots . p-1)$;

2 sets of p conjugate subgroups of order pq, $\{s, u\}$, $\{t, u\}$;

1 self-conjugate not cyclical subgroup of order p^2, $\{s, t\}$.

The operations are distributed according to their orders as in the preceding case.

β). $\dfrac{p+1}{q}$ *an integer.*

In this case the $p+1$ subgroups of order p divide in respect to transformation by u into $\dfrac{p+1}{q}$ sets of q groups each. The q groups of any set are either all unchanged by u, or are permuted cyclically. The former assumption would lead to the group β_3, since $p-1$ is not divisible by q. (The case $q=2$ is considered

below.) Consequently, to obtain a new type we must assume that u transforms any subgroup of order p, as $\{s\}$, into a different subgroup of order p, as $\{t\}$,

$$u^{-1}su = t.$$

Also, the order of u being $q > 2$, we must take

$$u^{-2}su^2 = u^{-1}tu = s^\lambda t^\mu, \quad (\lambda \neq 0, \mu \neq 0),$$

where λ and μ must be so chosen that

$$u^{-q}su^q = u^{-(q-2)}(s^\lambda t^\mu)\, u^{q-2} = s.$$

As no subgroup of order p is transformed into itself by any operation of order q, there is no self-conjugate subgroup of order p, and for the same reason, no subgroup of order pq.

This group β_7 exists only if $p+1$ is divisible by q. Also the case $q = 2$ is inadmissible, since we must then have

$$u^{-1}su = t, \; u^{-1}tu = s,$$

and therefore

$$u^{-1}(st)\, u = st,$$

so that u would transform one and therefore two groups of order p into themselves, and we should have β_3 or β_4. The group has only two generators connected by the identity

$$(u^{-1}su)^{-1}s\,(u^{-1}su) = s \text{ or } u^{-1}s^{-1}usu^{-1}su = s.$$

The group contains

p^2 conjugate subgroups of order q, $(s^\alpha t^\beta)^{-1}u\,(s^\alpha t^\beta)$, $(\alpha, \beta = 0, 1, \ldots . p-1)$;

$\dfrac{p+1}{q}$ sets of q conjugate subgroups of order p; no subgroup of order pq;

1 self-conjugate subgroup of order p^2, $\{s, t\}$.

Its operations are of the same orders as in the two preceding cases.

The lowest order for this type is $5^2.3 = 75$. A substitution group of this type and order is generated by

$s = (1, 2, 3, 4, 5)\,(6, 7, 8, 9, 10)\,(11, 12, 13, 14, 15)(16, 17, 18, 19, 20)$
$(21, 22, 23, 24, 25)(26, 27, 28, 29, 30)(31, 32, 33, 34, 35)(36, 37, 38, 39, 40)$
$(41, 42, 43, 44, 45)(46, 47, 48, 49, 50)(51, 52, 53, 54, 55)(56, 57, 58, 59, 60)$
$(61, 62, 63, 64, 65)(66, 67, 68, 69, 70)(71, 72, 73, 74, 75)$,

$u = (1, 26, 51)(2, 31, 75)(3, 36, 69)(4, 41, 63)(5, 46, 57)(6, 50, 52)(7, 30, 71)$
$(8, 35, 70)(9, 40, 64)(10, 45, 58)(11, 44, 53)(12, 49, 72)(13, 29, 66)(14, 34, 65)$
$(15, 39, 59)(16, 38, 54)(17, 43, 73)(18, 48, 67)(19, 28, 61)(20, 33, 60)$
$(21, 32, 55)(22, 37, 74)(23, 42, 68)(24, 47, 62)(25, 27, 56)$.

§5.

The Groups of Order pq^2, $p > q$.

The application of Sylow's theorem shows that

Every group of order pq^2, the case $p=3$, $q=2$ excepted, contains a single and therefore self-conjugate subgroup of order p, and in every case either 1 *or p subgroups of order q^2.*

The self-conjugate subgroup of order p we designate throughout by

$$H_p \equiv \{s\} \equiv (1, s, s^2, \ldots . s^{p-1}).$$

The subgroups of order q^2 being either cyclical or not cyclical, there are four general cases to be discussed.

A.—*A single and therefore self-conjugate H_{q^2}.*

a). *The H_{q^2} cyclical*; $H_{q^2} \equiv \{t\} \equiv (1, t, t^2, \ldots . t^{q^2-1})$.

The group $\{t\}$ being self-conjugate, we have

$$s^{-1}ts = t^{\mu}, \quad (\mu^p \equiv 1 \bmod q^2, \ \therefore \ \mu = 1).$$

Accordingly, s and t being commutative, st is of order pq^2, and the group is cyclical.

This group γ_1 contains 1 self-conjugate cyclical subgroup of each of the orders q, p, pq, q^2. Its operations are distributed according to their orders as follows:

Order ...	1	q	p	pq	q^2	pq^2
Number ..	1	$q-1$	$p-1$	$(p-1)(q-1)$	$q(q-1)$	$(p-1)(q-1)q$

b). *The H_{q^2} not cyclical*; $H_{q^2} \equiv \{t, u\}$, $(tu = ut)$.

Since $\{s\}$ is self-conjugate, we must have

$$t^{-1}st = s^{\mu}, \qquad u^{-1}su = s^{\nu},$$

$$\therefore \quad s^{-1}ts = ts^{1-\mu}, \quad s^{-1}us = us^{1-\nu}.$$

But, as $\{t, u\}$ is also self-conjugate, it follows that $\mu = \nu = 1$. The identities are therefore

$$t^{-1}st = s, \quad u^{-1}su = s, \quad t^{-1}ut = u,$$

and the operations of the group are all commutative.

The operation s in combination with the operations $t, u, tu, tu^2, \ldots . tu^{q-1}$ generates $q + 1$ cyclical groups of order pq. These groups having the power of s in common, contain beside these $(q + 1)p(q - 1) = p(q^2 - 1)$ different operations, which with the powers of s make up the entire group.

This group γ_2 contains

$q + 1$ self-conjugate subgroups of order q, $\{t\}$, $\{ut^a\}$, $(a = 0, 1, \ldots . q - 1)$;
1 self-conjugate subgroup of order p, $\{s\}$;
$q + 1$ self-conjugate cyclical subgroups of order pq, $\{s, t\}$, $\{s, ut^a\}$,
$(a = 0, 1, \ldots . q - 1)$;
1 self-conjugate not cyclical subgroup of order q^2, $\{u, t\}$.

The operations are distributed according to their orders as follows:

Order....	1	q	p	pq
Number..	1	$q^2 - 1$	$p - 1$	$(p-1)(q^2-1)$

A substitution group of order 12 of this type is generated by

$$s = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12),$$
$$t = (1, 4)(2, 5)(3, 6)(7, 10)(8, 11)(9, 12),$$
$$u = (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12).$$

B.—*p conjugate H_{q^2}'s; $\frac{p-1}{q}$ an integer.*

a). The H_{q^2}'s cyclical.

Denoting any one of the H_{q^2}'s by $\{t\}$, we must have

$$t^{-1}st = s^\mu, \quad (\mu^{q^2} \equiv 1 \bmod p).$$

Here $\mu \neq 1$, for $\mu = 1$ would lead to the group γ_1. The p H_{q^2}'s are therefore obtained by transforming $\{t\}$ with respect to s. They are, apart from their order of sequence,

$$\{t\}, \ \{ts\}, \ \{ts^2\}, \ \ldots . \ \{ts^{p-1}\}.$$

There are now two cases to be distinguished, according as μ is a primitive root of the congruence $\mu^q \equiv 1 \pmod p$, or of the congruence $\mu^{q^2} \equiv 1 \pmod p$.

In the former case t^q is commutative with s. The powers of t^q are common to all the H_{q^2}'s. These powers combined with s generate a self-conjugate cyclical subgroup of order pq. The pq operations of the latter and the $p(q^2-q)$ operations of order q^2 in the H_{q^2}'s make up the entire group.

This group γ_3 exists only when $\dfrac{p-1}{q}$ is an integer. Its identity is

$$t^{-1}st = s^{\mu}, \quad (\mu^q \equiv 1 \bmod p).$$

It contains

1 self-conjugate subgroup of order q, $\{t^q\}$;
1 self-conjugate subgroup of order p, $\{s\}$;
1 self-conjugate cyclical subgroup of order pq, $\{s, t^q\}$;
p conjugate cyclical subgroups of order q^2, $\{s^{-a}ts^a\}$, $(a = 0, 1, \ldots . p-1)$.

Its operations are distributed according to their orders as follows:

Order....	1	q	p	pq	q^2
Number..	1	$q-1$	$p-1$	$(p-1)(q-1)$	$pq(q-1)$

A substitution group of order 12 of this type is generated by

$$s = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12),$$
$$t = (1, 4, 7, 10)(2, 6, 8, 12)(3, 5, 9, 11).$$

In the second case, μ is a primitive root of the congruence $\mu^{q^2} \equiv 1 \pmod p$. In this case the H_{q^2}'s differ in all their operations except identity. Their operations of order q combine with s to generate a not cyclical group of order pq.

This group γ_4 exists only if $\dfrac{p-1}{q^2}$ is an integer. Its identity is

$$t^{-1}st = s^{\mu}, \quad (\mu^{q^2} \equiv 1 \bmod p, \ \mu^q \not\equiv 1 \bmod p).$$

The group contains

p conjugate subgroups of order q, $\{s^{-a}t^q s^a\}$, $(a = 0, 1, \ldots . p-1)$;
1 self-conjugate subgroup of order p, $\{s\}$;
1 self-conjugate not cyclical subgroup of order pq, $\{s, t^q\}$;
p conjugate cyclical subgroups of order q^2, $\{s^{-a}ts^a\}$, $(a = 0, 1, \ldots . p-1)$.

Its operations are distributed according to their orders as follows:

Order....	1	q	p	q^2
Number..	1	$p(q-1)$	$p-1$	$pq(q-1)$

A substitution group of order 20 of this type is generated by

$$s=(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)(16, 17, 18, 19, 20),$$
$$t=(1, 6, 11, 16)(2, 8, 15, 19)(3, 10, 14, 17)(4, 7, 13, 20)(5, 9, 12, 18).$$

The groups γ_3 and γ_4 are analogous to the not cyclical type of order pq, the groups of order q in the latter being here replaced by the cyclical groups of order q^2.

b). The H_{q^2}'s not cyclical.

Any one of the H_{q^2}'s being denoted by $\{t, u\}$, we have

$$t^{-1}st=s^{\mu}, \quad u^{-1}su=s^{\nu}.$$

Here μ and ν cannot both be equal to 1, for this would give the group γ_2. We have two cases to consider, according as one or both of them is different from 1.

In the first case, suppose $\mu=1$, $\nu \neq 1$. Then

$$t^{-1}st=s, \quad u^{-1}su=s^{\nu};$$
$$\therefore\ s^{-1}ts=t, \quad s^{-1}us=us^{1-\nu}.$$

The p H_{q^2}'s arise by transformation of $\{t, u\}$ with respect to s. These p groups, having the subgroup $\{t\}$ in common, contain beside this $p(q^2-q)$ different operations. The combination of s with t furnishes a cyclical group of order pq, the pq operations of which complete the entire group.

This group γ_5 exists only if $\dfrac{p-1}{q}$ is an integer. Its identities are

$$t^{-1}ut=u, \quad t^{-1}st=s, \quad u^{-1}su=s^{\nu}, \quad (\nu \neq 1).$$

The group contains

1 self-conjugate subgroup of order q, $\{t\}$;
q sets of p conjugate subgroups of order q;
$\{s^{-\alpha}(ut^{\beta})s^{\alpha}\}$, $(\alpha=0, 1, \ldots. p-1, \beta=1, 2, \ldots. q-1)$;
1 self-conjugate subgroup of order p, $\{s\}$;

1 self-conjugate cyclical subgroup of order pq, $\{s, t\}$;
p conjugate not cyclical subgroups of order q^2, $\{t, s^{-a}us^a\}$, $(a = 0, 1, \ldots . p-1)$.

The operations are distributed according to their orders as follows:

Order....	1	q	p	pq
Number..	1	$(pq+1)(q-1)$	$p-1$	$(p-1)(q-1)$

A substitution group of order 12 of this type is generated by

$$s = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12),$$
$$t = (1, 4)(2, 5)(3, 6)(7, 10)(8, 11)(9, 12),$$
$$u = (1, 7)(2, 9)(3, 8)(4, 10)(5, 12)(6, 11).$$

Secondly, suppose that in the identities

$$t^{-1}st = s^{\mu}, \quad u^{-1}su = s^{\nu},$$

neither μ nor ν is equal to 1. If $\mu = \nu$, we have

$$(u^{-1}t)^{-1}s\,(u^{-1}t) = s,$$

which leads to the group γ_5. Accordingly we must take

$$t^{-1}st = s^{\mu}, \quad u^{-1}su = s^{\nu}, \quad (\mu \neq 1, \ \nu \neq 1, \ \mu \neq \nu);$$
$$\therefore \ s^{-1}ts = ts^{1-\mu}, \quad s^{-1}us = us^{1-\nu}.$$

As before, the $p\, H_{q^2}$'s arise by transformation of $\{t, u\}$ by s. These groups have now no operation except identity common to any two of them. Their $p(q^2-1)$ operations of order q with the p operations $\{s\}$ make up the entire group.

This group γ_6 exists only if $\dfrac{p-1}{q}$ is an integer. Also q must be > 2, for the congruence $\mu^2 \equiv 1 \pmod{p}$ has only one root different from 1. The identities are

$$u^{-1}tu = t, \ t^{-1}st = s^{\mu}, \ u^{-1}su = s^{\nu}, \quad (\mu \neq 1, \ \nu \neq 1, \ \mu \neq \nu).$$

The group contains

$q+1$ sets of p conjugate subgroups of order q,

$$\{s^{-a}ts^a\}, \ \{s^{-a}(t^{\beta}u)s^a\}, \quad (a = 0, 1, \ldots . p-1; \ \beta = 0, 1, \ldots . q-1);$$

1 self-conjugate subgroup of order p, $\{s\}$;
p conjugate not cyclical subgroups of order q^2, $s^{-a}\{t, u\}s^a$, $(a = 0, 1, \ldots . p-1)$.

The operations are distributed according to their orders as follows:

Order....	1	q	p
Number..	1	$p(q^2-1)$	$p-1$

A substitution group of order 63 of this type is generated by

$s=(1, 2, 3, 4, 5, 6, 7)(8, 9, 10, 11, 12, 13, 14)(15, 16, 17, 18, 19, 20, 21)$
$(22, 23, 24, 25, 26, 27, 28)(29, 30, 31, 32, 33, 34, 35)(36, 37, 38, 39, 40, 41, 42)$
$(43, 44, 45, 46, 47, 48, 49)(50, 51, 52, 53, 54, 55, 56)(57, 58, 59, 60, 61, 62, 63).$
$t=(1, 8, 15)(2, 10, 19)(3, 12, 16)(4, 14, 20)(5, 9, 17)(6, 11, 21)(7, 13, 18)$
$(22, 29, 36)(23, 31, 40)(24, 33, 37)(25, 35, 41)(26, 30, 38)(27, 32, 42)(28, 34, 39)$
$(43, 50, 57)(44, 52, 61)(45, 54, 58)(46, 56, 62)(47, 51, 59)(48, 53, 63)(49, 55, 60).$
$u=(1, 22, 43)(2, 26, 45)(3, 23, 47)(4, 27, 49)(5, 24, 44)(6, 28, 46)(7, 25, 48)$
$(8, 29, 50)(9, 33, 52)(10, 30, 54)(11, 34, 56)(12, 31, 51)(13, 35, 53)(14, 32, 55)$
$(15, 36, 57)(16, 40, 59)(17, 37, 61)(18, 41, 63)(19, 38, 58)(20, 42, 60)(21, 39, 62).$

This group and the preceding are again analogous to the not cyclical type of order pq, the subgroups of order q in the latter being here replaced by the not cyclical subgroups of order p^2.

C.—*The special group of order* 12.

For the order 12, Sylow's theorem shows the possibility of an exceptional group with *four* conjugate subgroups of order 3. These four subgroups contain 8 operations of order 3. The 4 remaining operations must then form a self-conjugate subgroup of order 4.

The latter cannot be cyclical; for if we denote it by

$$1, t, t^2, t^3$$

and any operation of order 3 by s, we should have

$$s^{-1}ts=t, \quad (a^3\equiv 1 \bmod 4).$$

Then $a=1$, and st would be of order 12, so that the G_{12} would be cyclical.

The exceptional group of order 12 therefore contains 8 operations of order 3, 3 of order 2, and identity. If we denote the operations of order 2 by t, u,

ut, and any operation of order 3 by s, then s transforms t, u, ut in a cycle. For if s left t unchanged, st would be of order 6. Accordingly we have the identities

$$u^{-1}tu=t, \quad s^{-1}ts=u, \quad s^{-1}us=ut,$$

from which the group is fully determined.

This group is in fact isomorphic with the alternating group of 4 letters.

§6.

The Groups of Order pqr; $(p>q>r)$.

From Sylow's theorem a group G of order pqr contains either 1 or qr subgroups of order p.

In the latter case these qr subgroups would contain $qr\,(p-1)$ distinct operations of order p, leaving only qr operations of G to be determined. Among these qr operations there must be at least one, t, of order q, and at least one, u, of order r. If t transforms u into itself, then t and u generate a cyclical group of order qr, which exactly supplies the qr missing operations. This group contains only a single subgroup, $\{t\}$, of order q, which is therefore the only subgroup of this order contained in G. Again, if t does not transform u into itself, then it transforms the group $\{u\}$ into q conjugate groups of order r. These contain $q\,(r-1)$ distinct operations of order r, leaving only the q powers of t. In this case also, then, the group G contains only one subgroup, $\{t\}$, of order q.

Now, if s is any operation of order p contained in G, then s must transform t into itself. Accordingly, s and t generate a cyclical group H of order pq. This group contains all the operations of order p, q or pq which occur in G. For if τ is any such operation, contained in G but not in H, then the operations

$$H, \ \tau H, \tau^2 H, \ldots . \tau^{q-1} H$$

will all be different. But their number is $pq^2>pqr$. But H contains only one subgroup $\{s\}$ of order p, which is therefore the only subgroup of order p contained in G.

Accordingly, a group of order pqr contains only one subgroup of order p.

If the subgroup of order p is $\{s\}$, and if t is any operation of order q contained in G, then t must transform s into one of its powers. Consequently, s and t generate a group of order pq. By the same reasoning employed above, this

group contains all the operations of order p, q, or pq that occur in G; it is therefore self-conjugate.

Every group G of order pqr contains a self-conjugate subgroup H of order pq. The latter contains all the operations of order p, q, or pq that occur in the entire group.

We have now to distinguish two principal cases according as the subgroup H of order pq is cyclical or not cyclical. Throughout we use s, t, and u, as heretofore, to designate operations of orders p, q, and r.

A.—*The subgroup H cyclical*; $st = ts$.

This case can occur for all values of p and q, and is the only possibility if q is not a divisor of $p-1$.

Since G contains here only one subgroup of order q, as well as only one of order p, we must have

$$u^{-1}su = s^{\mu}, \quad u^{-1}tu = t^{\nu}.$$

There are four subcases, according as

1) $\mu = 1, \nu = 1$; 2) $\mu = 1, \nu \neq 1$; 3) $\mu \neq 1, \nu = 1$; 4) $\mu \neq 1, \nu \neq 1$.

1) In this case, s, t and u being all commutative, the operation stu is of order pqr, and G is cyclical. This group, δ_1, contains one subgroup of each of the orders p, q, r, pq, pr, and qr; and these subgroups are all cyclical and self-conjugate. The operations of δ_1 are distributed according to their orders as follows:

Order...	1	p	q	r	pq	pr	qr	pqr
Number.	1	$p-1$	$q-1$	$r-1$	$(p-1)(q-1)$	$(p-1)(r-1)$	$(q-1)(r-1)$	$(p-1)(q-1)(r-1)$

2). This case requires that r should be a divisor of $p-1$. The operations u and t generate a not cyclical group of order qr, in which the q subgroups of order r are

$$\{t^{-a}ut^{a}\}, \quad (\alpha = 0, 1, \ldots . q-1).$$

The operations $t^{-a}ut^{a}$ are all commutative with s. Each of them, taken with s, generates a cyclical group of order pr. These q groups have only the powers of s in common; beside these they contain $q(pr-p)$ distinct operations, which with the operations of H make up the entire group.

This group δ_2 exists only if r is a divisor of $p-1$. Its identities are $st=ts$, $su=us$, $u^{-1}tu=t^{\mu}$. It contains

1 self-conjugate subgroup of order p, $\{s\}$;
1 self-conjugate subgroup of order q, $\{t\}$;
q conjugate subgroups of order r, $\{t^{-\alpha}ut^{\alpha}\}$, $(\alpha=0, 1, \ldots . q-1)$;
1 self-conjugate cyclical subgroup of order pq, $\{s, t\}$;
1 self-conjugate not cyclical subgroup of order qr, $\{t, u\}$;
q conjugate cyclical subgroups of order pr, $\{s, t^{-\alpha}ut^{\alpha}\}$.

The operations of the group are distributed according to their orders as follows:

Order ..	1	p	q	r	pq	pr
Number.	1	$p-1$	$q-1$	$q(r-1)$	$(p-1)(q-1)$	$q(p-1)(r-1)$

A substitution group of this type of order 30 is generated by

$s=(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)$
$(16, 17, 18, 19, 20)(21, 22, 23, 24, 25)(26, 27, 28, 29, 30)$.
$t=(1, 6, 11)(2, 7, 12)(3, 8, 13)(4, 9, 14)(5, 10, 15)$
$(16, 21, 26)(17, 22, 27)(18, 23, 28)(19, 24, 29)(20, 25, 30)$.
$u=(1, 16)(2, 17)(3, 18)(4, 19)(5, 20)$
$(6, 26)(7, 27)(8, 28)(9, 29)(10, 30)$
$(11, 21)(12, 22)(13, 23)(14, 24)(15, 25)$.

3). This case differs from 2) only in the exchange of the roles of s and t.

4). In this case r must be a divisor of both $p-1$ and $q-1$. The operations of the group can then be written as follows (where $\sigma=st$):

$$
\begin{array}{lllll}
1 & , \sigma & , \sigma^2 & , \ldots . \sigma^{pq-1} & , \\
u & , \sigma^{-1}u\sigma & , \sigma^{-2}u\sigma^2 & , \ldots . \sigma u\sigma^{-1} & , \\
u^2 & , \sigma^{-1}u^2\sigma & , \sigma^{-2}u^2\sigma^2 & , \ldots . \sigma u^2\sigma^{-1} & , \\
\ldots & \ldots & \ldots & \ldots & \\
u^{r-1}, & \sigma^{-1}u^{r-1}\sigma, & \sigma^{-2}u^{r-1}\sigma^2, & \ldots . \sigma u^{r-1}\sigma^{-1}. &
\end{array}
$$

For these operations are all different, since

$$\sigma^{-i}u^{\lambda}\sigma^{i}=\sigma^{-j}u^{\mu}\sigma^{j}$$

would require
$$\sigma^{-(i-j)}u^{\lambda}\sigma^{i-j}=u^{\mu},$$

where μ must be $=\lambda$. But then, a power of σ being commutative with u, either s or t would be commutative with u, which is excluded.

This group δ_4 is therefore an analogue of the not cyclical type of order pq. It exists only if r is a divisor of both $p-1$ and $q-1$. Its identities are

$$st=ts,\quad u^{-1}su=s^{\mu},\quad u^{-1}tu=t^{\nu}.$$

It contains

1 self-conjugate subgroup of order p, $\{s\}$;
1 self-conjugate subgroup of order q, $\{t\}$;
pq conjugate subgroups of order r, $\{(st)^{\alpha}u\,(st)^{\alpha}\}$, $(\alpha=0,1,2,\ldots.pq-1)$;
1 self-conjugate cyclical subgroup of order pq, $\{st\}$;
q conjugate not cyclical subgroups of order pr, $\{s,\,t^{-\alpha}ut^{\alpha}\}$;
p conjugate not cyclical subgroups of order qr, $\{t,\,s^{-\alpha}us^{\alpha}\}$.

Its operations are distributed according to their orders as follows:

Order ..	1	p	q	r	pq
Number.	1	$p-1$	$q-1$	$pq(r-1)$	$(p-1)(q-1)$

A substitution group of this type of order 30 is generated by

$s=(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)$
$(16, 17, 18, 19, 20)(21, 22, 23, 24, 25)(26, 27, 28, 29, 30)$.
$t=(1, 6, 11)(2, 7, 12)(3, 8, 13)(4, 9, 14)(5, 10, 15)$
$(16, 21, 26)(17, 22, 27)(18, 23, 28)(19, 24, 29)(20, 25, 30)$.
$u=(1, 16)(2, 20)(3, 19)(4, 18)(5, 17)$
$(6, 26)(7, 30)(8, 29)(9, 28)(10, 27)$
$(11, 21)(12, 25)(13, 24)(14, 23)(15, 22)$.

B.—*The subgroup H not cyclical*; $t^{-1}st=s^{\mu}$, $(\mu \neq 1)$.

This is possible only if q is a divisor of $p-1$. The group H now contains one subgroup of order p, and p subgroups of order q. Any operation u of order

r must transform at least one of these q subgroups, say $\{t\}$, into itself. We have then, as under A,

$$u^{-1}su = s^{\nu}, \quad u^{-1}tu = t^{\rho}.$$

It follows that

$$u^{-1}(t^{-1}st)\,u = u^{-1}(s^{\mu})\,u = s^{\mu\nu}.$$

But, on the other hand,

$$\begin{aligned} u^{-1}(t^{-1}st)\,u &= u^{-1}t^{-1}u\,.\,u^{-1}su\,.\,u^{-1}tu \\ &= t^{-\rho}.s^{\nu}.t^{\rho} \\ &= s^{\mu^{\rho}\nu}. \end{aligned}$$

Hence we must have

$$\mu^{\rho}\nu \equiv \mu\nu \pmod{p},$$
$$\mu^{\rho-1} \equiv 1 \pmod{p}.$$

But since $\rho - 1 < q$ and $\mu \neq 1$, this is only possible if

$$\rho = 1.$$

We have therefore

$$u^{-1}su = s^{\nu}, \quad u^{-1}tu = t,$$

and there are two cases to be distinguished according as 5) $\nu = 1$ or 6) $\nu \neq 1$.

5). Here t and u generate a cyclical group of order qr which s transforms into p distinct conjugate groups of this order. These groups have the powers of u in common; beside these they contain $p\,(qr - r)$ distinct operations, which with the powers of su make up the entire group. It is readily seen that this group is of essentially the same form as 2) and 3) under A, the operations u, s, t here playing the same role as s, t, u in 2).

6). In this case r as well as q must be a divisor of $p - 1$. As in 5), the operations t and u generate a cyclical group of order qr which s transforms into p conjugate groups of this order. But in the present case these groups have no operation, except identity, in common. They contain therefore $p\,(qr - 1)$ distinct operations, which with the powers of s make up the entire group.

This group δ_5 exists only if q and r are both divisors of $p - 1$. Its identities are

$$t^{-1}st = s^{\mu}, \quad u^{-1}su = s^{\nu}, \quad u^{-1}tu = t.$$

It contains

1 self-conjugate subgroup of order p, $\{s\}$;
p conjugate subgroups of order q, $\{s^{-a}ts^{a}\}$, $(a = 0, 1, \ldots . p - 1)$;
p conjugate subgroups of order r, $\{s^{-a}us^{a}\}$, $(a = 0, 1, \ldots . p - 1)$;

1 self-conjugate not cyclical subgroup of order pq, $\{s, t\}$;
1 self-conjugate not cyclical subgroup of order pr, $\{s, u\}$;
p conjugate cyclical subgroups of order qr, $\{s^{-a}ts^{a}, u\}$, $(a = 0, 1, \ldots . p-1)$.

Its operations are distributed according to their orders as follows:

Order...	1	p	q	r	qr
Number.	1	$p-1$	$p(q-1)$	$p(r-1)$	$p(q-1)(r-1)$

A substitution group of this type of order 42 is generated by

$s = (1, 2, 3, 4, 5, 6, 7)(8, 9, 10, 11, 12, 13, 14)$
$(15, 16, 17, 18, 19, 20, 21)(22, 23, 24, 25, 26, 27, 28)$
$(29, 30, 31, 32, 33, 34, 35)(36, 37, 38, 39, 40, 41, 42)$.
$t = (1, 8, 15)(2, 10, 19)(3, 12, 16)(4, 14, 20)(5, 9, 17)$
$(6, 11, 21)(7, 13, 18)(22, 29, 36)(23, 31, 40)(24, 33, 37)$
$(25, 35, 41)(26, 30, 38)(27, 32, 42)(28, 34, 39)$.
$u = (1, 22)(2, 28)(3, 27)(4, 26)(5, 25)(6, 24)(7, 23)$
$(8, 29)(9, 35)(10, 34)(11, 33)(12, 32)(13, 31)(14, 30)$
$(15, 36)(16, 42)(17, 41)(18, 40)(19, 39)(20, 38)(21, 37)$.

ANN ARBOR, *Sept.*, 1892.

The Nature and Effect of Singularities of Plane Algebraic Curves.

By Charlotte Angas Scott.

1. In Vol. XIV of this Journal I gave an account of a geometrical method of analysing Higher Singularities, by means of which there may be found for any singularity a penultimate form involving a series of nodes with a certain number of evanescent loops. It was there stated that the method is directly applicable, in general, to the determination only of the point components of the singularity, though in certain cases it determines the inflexions. I propose now to remove this restriction, showing that the process enables us, in every case, to enumerate the *double lines* (double tangents and inflexional tangents) involved in the singularity.

The definitions usually adopted for double points and double lines, as occurring in composition, are purely algebraic, $\varkappa$ (the number of cusps) being by definition one less than the L. C. D. of the exponents in a cyclic system of expansions, and ν (the number of nodes) being defined for any singularity, simple or compound, by equating $2\nu + 3\varkappa$ to the discriminantal index; it appears at once that these definitions agree with the ordinary ones for the simple node and cusp; it is then shown that ν, $\varkappa$, being so defined for any singularity, Plücker's equations hold. The numbers τ, ι are similarly defined by means of the line equation. See H. J. S. Smith, "On the Higher Singularities of Plane Curves," 1873–6 (Proc. Lond. Math. Soc., VI, 153), and Brill, "Ueber Singularitäten ebener Curven," 1879 (Math. Ann., XVI, 348).

But an algebraic curve has a geometric existence, and the process here used for the analysis of singularities is a geometric process depending entirely on a geometric conception of singularities, even though the language used is necessarily algebraic. It seems, therefore, advisable to attempt a harmonious geometric treatment of singularities (based on Plücker's views, given in his Theorie der

Algebraischen Curven, pp. 200–207, here reproduced in substance in sections 2, 3, 6, 9, 10), depending on precise geometric definitions, making use of a geometric method of analysis, and leading to a natural classification of singularities that shall give due weight to the point and line conceptions. It will then be necessary to prove that Plücker's equations hold for singularities thus defined; this is only partially accomplished in the present paper. The various relations proved by H. J. S. Smith and Brill present themselves, expressing simple geometric facts.

2. The object of our investigation is a certain geometric form, a *curve*. We define this by means of an equation connecting the coordinates of any point on it; this amounts to considering the curve as traced by a point moving subject to the law expressed by the equation. But we may also define the curve by means of an equation connecting the coordinates of any tangent; this amounts to considering the curve as enveloped by a line moving subject to the law expressed by this equation. The curves with which we are concerned are those only for which these laws are expressed by rational integral algebraic equations. Adopting the first view, we may consider the curve as the limit of a polygon defined by its vertices; adopting the second view, the polygon is regarded as defined by its sides. We have thus the dual conception of the curve, as described by a point, enveloped by a line, "the point moving continuously along the line, the line rotating continuously about the point" (Plücker).

3. Now on an arc so considered, it follows from the definition that the line elements may be derived from the point elements by joining each to the next; and that the point elements may be derived from the line elements by marking the intersection of each with the next. Consequently *either* equation, point or line, will yield us all the properties of the arc.

But suppose it happens that either part of the characterisation becomes incorrect? It may happen that the point gives up its continuous motion without any effect on the continuous rotation of the line; the point, that is, "attains a limiting position and then moves in the contrary direction" (p. 202). This we naturally expect to have an effect on the point equation, but we cannot expect any trace of it in the line equation. Again, says Plücker, this singularity may occur in the motion of the line; we do not expect to discover this if we confine ourselves to the point equation. And in the third place, these singu-

larities may occur simultaneously. Note then that in the first case we must not expect the tangents to be derived by joining each point to the next; and in the second case, the points cannot be derived by marking the intersection of the tangent with the next.

Plücker's view is then that at the simplest point singularity, the cusp, the tracing point travels along the tangent till it attains a certain position O, and then turns back along the tangent. Now the elementary theory of maxima and minima teaches us to regard a maximum or minimum as a stationary value or position, characterised by the coincidence of two consecutive elements (of magnitude or position). There being then a cusp at O, we may consider the tracing point as occupying the position O *twice*, so that representing its course symbolically by PO_1O_2Q, consecutive positions of the enveloping line are PO_1, O_2Q; i. e. PO, OQ (Fig. 1). Note in the first place that the "tangents at a cusp" are

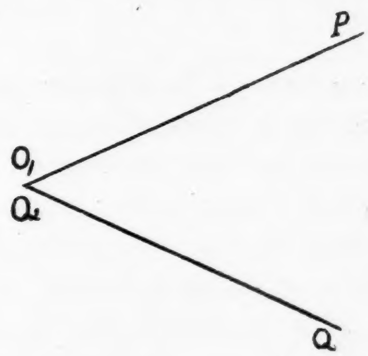

FIG. 1.

consecutive, not *coincident*, for coincident tangents cause a *line* singularity; and in the second place, that the line O_1O_2 is *not in any sense a tangent*. This may be further elucidated by comparison with the reciprocal singularity, the simple inflexion; the two points of contact of an inflexional tangent are consecutive, not coincident; and we do not regard *any* point on the inflexional tangent as a point of the curve.

4. De Morgan (1856, Camb. Phil. Trans., IX, part 4) appears inclined to adopt the view that at a cusp dy/dx has an infinite number of coexisting values. He states this possibility with reference to the evanescent oval; but the context

suggests that he has also the evanescent loop in mind. This would agree with the view that the tangent at a cusp makes a half turn, the point suffering no singularity in its motion. Now the accepted penultimate form for a cusp, the evanescent loop (Fig. 2), appears to justify this; the tangent does turn through 180°, and the motion of the point along the tangent is unaffected. But if we accept this view for the cusp, we must accept the reciprocal view for the

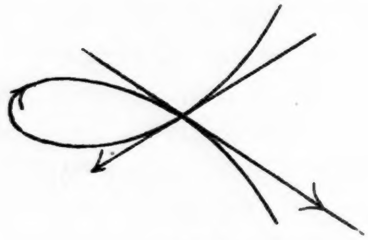

FIG. 2.

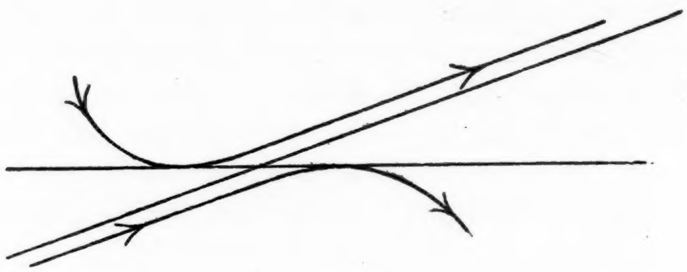

FIG. 3.

inflexion. In the penultimate form we have an asymptote inclined at a vanishing angle to a double tangent whose points of contact are infinitely near together (Fig. 3); thus if we account for the cusp by means of a sudden half turn of the tangent, we must account for the inflexion by means of a sudden flight through infinity, by which half a complete line is described.* Now this explanation ensures the proper zero values for $ds/d\phi$ and $d\phi/ds$ at the cusp and inflexion; but it makes the inflexion a point singularity, and the cusp a line singularity, contrary to what we know to be the case. Adopting this view really amounts to holding on to the penultimate form instead of taking the last step, which, as usual, introduces a discontinuity, shown here by the rejection of a linear factor from the reciprocal equation; this factor representing, when we form the line equation of a curve with a cusp at O, the point O, and when we form the point equation of a curve with an inflexional tangent ω, the line ω. It involves our confining ourselves to curves without cusps or inflexions, which, of course, do not exist (as proper curves) for an order greater than 2.

* I here adopt the view reciprocal to that in which the angle about a point is considered as four right angles, and regard the complete line as consisting of the line taken twice, the upper "edge" continuous with the lower "edge," and therefore two passages through infinity being required for the complete description of the line.

As an additional example of the care needed in drawing conclusions as to the line element from the point diagram or the point equation, and vice versa, consider the tacnode formed by the simple contact of two branches, properly described as having in common one point and the tangent there, a description which is reciprocal to itself. The point equation expresses this "the branches have two consecutive points common," the line equation says "the branches have two consecutive lines common." But if we combine these and say the branches have two points and two tangents common, we are giving what is properly the description of the oscnode. Fig. 4 represents the point and line elements of

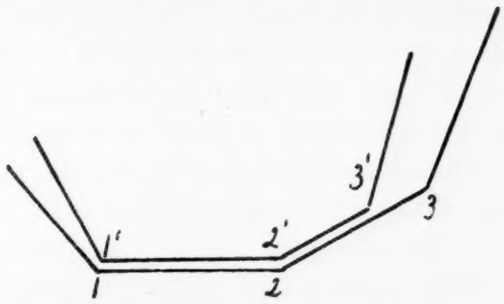

FIG. 4.

a tacnode; $1 = 1'$ and $2 = 2'$ are the common points; $12 = 1'2'$ is one of the common tangents; and if we consider the line elements before 1 as *not* coincident, (therefore, by continuity, necessarily consecutive) then we must consider 23, $2'3'$ as coincident; but 3 and $3'$ must be regarded as consecutive, not as coincident points.

5. These examples illustrate the fact that the expression by means of points is a convention, and the expression by means of lines a different convention; and as these, or the penultimate forms derived from them, may express different parts of the truth, we cannot expect to be able to deduce the whole truth from one expression. The penultimate form in each of the diagrams used in the method of inversion is derived from the point equation, and must therefore be cautiously used in drawing conclusions as to line singularities; we have no grounds for assuming it to be the penultimate line form, and in fact the one diagram for the resolution of the simple cusp shows that this is not the case.

Further, the single algebraic equation is deceptive in another way, in that it does not discriminate between *consecutive* and *coincident.* Thus e. g. the point equation for the tangents at a cusp gives coincident lines, whereas we have seen that they are really consecutive; the tangents at a tacnode or at a cusp of the second species are properly coincident.

6. When we consider a curve, then, as defined by its point equation, cusps and nodes present themselves as singularities; when by its line equation, inflexional and double tangents are the singularities; when however we use the intrinsic equation, cusps and inflexions are the singularities. Plücker regards nodes and double tangents as accidents of position, and not real *singularities*; stating that he confines himself to singularities on a single arc. But we shall find that the occurrence of more than one proper singularity at a point, even on a single arc, involves the occurrence of nodes and double tangents; e. g. two cusps cannot exist without a node; a cusp and inflexion induce at least one node and one double tangent. We shall therefore frame our definition of singularities so as to include these. Plücker defines a singularity as caused by the turning back of the element; if this occurs twice, three times, etc., we have at the point two, three, cusps or inflexions. For purely geometric purposes it is perhaps more convenient to express this in terms of successive elements, instead of in terms of a single moving element; and this has the further convenience of enabling us to include nodes and double tangents.

It is occasionally convenient to adopt a symbolic notation; we use letters for positions in the plane, and when the describing element occupies a position more than once, we use suffixes, consecutive or not, as the case may be. Thus the symbol for a cusp is PO_1O_2Q, or $PO_{12}Q$; for an inflexion, po_1o_2q or $po_{12}q$; for a loop, $PO_1Q....RO_{n+2}S$; and for an evanescent loop, $\underset{n=0}{L} PO_1Q....RO_{n+2}S$.

7. We therefore formulate our conception of the singularities of a curve as follows: Any arc of the curve may be considered as yielding a succession of point elements occupying certain of the point positions in the plane; and a succession of line elements, occupying certain of the line positions in the plane; the coincidence of two point elements gives a double point, which is a cusp if the elements are consecutive, otherwise a node; the coincidence of two line elements gives a double line, which is an inflexional tangent if the elements are consecutive,

otherwise a double tangent. If k point elements coincide at O, the lowest terms in the point equation are of degree k, and the singularity is said to be of order k; similarly if l line elements coincide, the lowest terms in the line equation are of degree l, and the singularity is said to be of class l. Using a rather elliptical expression, we may say that the singularity has a point equation of degree k, and a line equation of degree l. Suppose now we have at O k consecutive elements (symbolically, the point is $PO_1O_2\ldots .O_kQ$); we have then $k(k-1)/2$ coincidences, of which $k-1$ $(O_1-O_2,\ O_2-O_3,\ O_3-O_4,\ldots .O_{k-1}-O_k)$ are coincidences of consecutive elements; the number of cusps in this singularity is therefore, by definition, $=k-1$; and similarly we may have a singularity of class l in which there are $l(l-1)/2$ double lines, of which $l-1$ are inflexional tangents.

8. Thus the special characteristic of the cusp and inflexional tangent, which differentiates them from the node and double tangent, is the *consecutiveness* of the elements involved in the coincidence,* and it would be convenient to have a special term to express this. Now it is precisely this property of consecutiveness in the point elements that is signalised by the presence of a branch point when ordinary analytical geometry is regarded from the standpoint of the theory of functions. But there is a decided practical inconvenience in borrowing a term from one subject to use in another, for its region of applicability is apt to be different in the two subjects; and further, in this case, there is the strong objection to the term branch point—that it is not available for the line theory. We need in analytical geometry words to express the consecutiveness of point or line elements, these being (1) not coincident, as e. g. the consecutive points in which a proper tangent meets the curve; (2) coincident, as e. g. these points when the line is not a tangent, but a line through a cusp. *Conjunction* and *consecution*, though hardly expressive enough, might serve. Thus for a curve of class n with $\varkappa$ cusps, there are $n+\varkappa$ point conjunctions with reference to any point, of which n are relative, $\varkappa$ absolute; or we may say there are $\varkappa$ consecutions; and for a curve of order m with ι inflexions, there are $m+\iota$ line

*It is foreign to the design of this paper to discuss the purely algebraic treatment of the question; but to facilitate comparison it may be mentioned that it is easy to show from the expansions that a point describing a superlinear branch *in conformity with the algebraic law expressed by the integral equation* must pass from expansion to expansion at the superlinearity; i. e. the consecutive point is to be found on a different expansion.

conjunctions with reference to any line, of which m are relative, ι absolute; or we may say there are ι consecutions. And again, the simplest superlinearity of order (or class) 3 contains 3 point (or line) coincidences, of which 2 are consecutions.

9. Plücker then points out that his definitions give at once the ordinary values for the radius of curvature; the same holds good for the modified definitions here adopted. The curve is defined in terms of s and ϕ, and the radius of curvature is $ds/d\phi$. Note in passing that it is better not to regard s as a function of ϕ, for in so doing we should be led to make the tacit assumption that ϕ, our independent variable, increases continuously, which would hamper us in the treatment of line singularities.* We therefore regard s and ϕ as functions of an independent variable t, and we have $\rho = L\frac{ds}{dt} \Big/ \frac{d\phi}{dt}$. Thus at a single cusp, by definition, $ds/dt = 0$, $\therefore\ \rho = 0$; and at an inflexion, $d\phi/dt = 0$, and $\therefore\ \rho = \infty$; and similarly if there are $\varkappa$ cusps, ι inflexions, so that $\varkappa\,(= k - 1)$ and $\iota\,(= l - 1)$ differential coefficients vanish, we have

$$\rho = L\frac{d^k s}{dt^k}(\delta t)^k \Big/ \frac{d^l \phi}{dt^l}(\delta t)^l$$

Thus if $k < l$, $\rho = \infty$;
if $k > l$, $\rho = 0$;
and if $k = l$, ρ is determinable.

10. In referring to the geometrical representation, it will be convenient occasionally to use the term *path* for the (real) path of the tracing point to or from O; so that an ordinary linear branch is made up of two paths joining at O; and a superlinear branch is made up of two paths, with some sort of junction at O. These two paths, with whatever the point does at O, make the *arc*, this word being used to denote the complete branch, linear or superlinear. The appearance of an arc depends on the way the two paths PO, $O'Q$ which meet at O,

* The same thing may be noticed in the use of the ordinary Cartesian equation, where, regarding y as a function of x, we tacitly assume that x increases continuously. This is all right in the purely algebraic theory, but it is an interesting question how far the imaginary "bridges" from circuit to circuit so introduced really belong to the geometric curve. They have not the organic connexion with the curve that the imaginary branches at a singularity have.

and have necessarily the same or consecutive tangents, are placed, and not at all on the way the transition from O to O' is accomplished. There are therefore four possible varieties, as represented in Fig. 5.

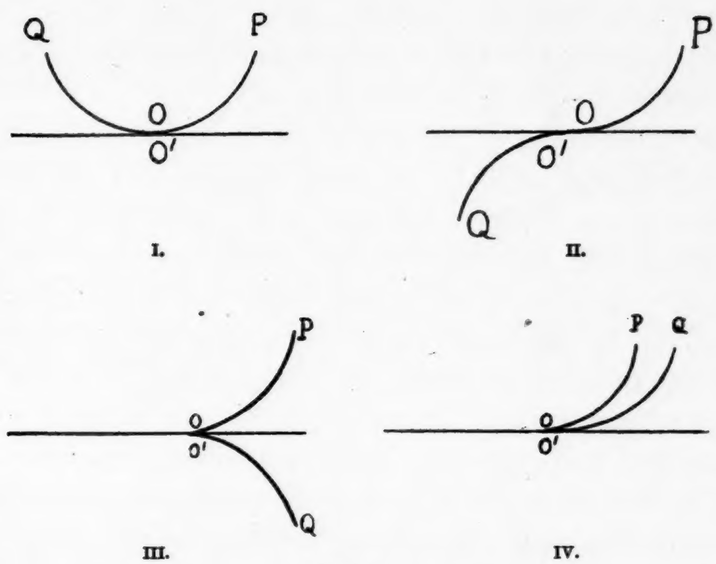

Fig. 5.

Now these depend only on k and l (Plücker, p. 207); for an even number of cusps (i. e. odd k) has no ultimate effect on the progress of the point; thus for I, II, k is odd; for III, IV, k is even; and an even number of inflexions (i. e. odd l) has no ultimate effect on the rotation of the line; thus for I, III, l is odd; and for II, IV, l is even.

But these arcs may be combined in any way, so producing composite singularities. From our definitions it follows that in a composite singularity the number of cusps and the number of inflexions are respectively the sums of those afforded by the separate superlinearities; but that the number of nodes and of double tangents will be augmented by the intersections and contacts of the complete branches. We shall speak in general of the singularity as containing δ double points (dps), of which ν are nodes, $\varkappa$ cusps; and λ double lines (dls), of which τ are double tangents and ι inflexional tangents.

11. The ordinary proofs show (or as in §7 it may be shown) that a singularity of order q and class r contains certainly $q(q-1)/2$ dps, and $r(r-1)/2$ dls.

But the simple example of a tacnode shows that the q-pt indicated by the equation may be but one of a series of consecutive multiple points; we shall therefore at present speak of a q-element, meaning one whose equation is of degree q, and which therefore contains certainly a q-point (line) and possibly other multiple points (lines) not yet apparent; say it contains a certain number of *latent* multiple points (lines). The main problem in the analysis of any singularity is to determine the latent double elements.

A process for determining the double points was explained in the article already referred to, Am. J., XIV, p. 301. Applied to a q-element, it clears away the obvious $q(q-1)/2$ coincidences, at the same time showing which of these are coincidences of consecutive points; for it represents the q points that come together at O by q points on the base line; and as the consecutiveness of points is preserved in the transformation, we have only to consider which of these q points are consecutive; thus e. g. $y^3 = x^4$ gives $y^3 = x$; the three points on the base are consecutive, we have therefore for the original singularity the symbol $PO_{123}Q$, and there are two cusps. But multiple points adjacent to O are unaffected by the transformation; thus the transformed equation presents to our view a certain number of q'-points—or rather elements—where $\Sigma q' \leqq q$, and these q'-elements must be further resolved. A superlinear branch in every case leads to a single q'-element, where $q' \leqq q$; and proceeding in this way, the singularity is gradually untwisted. Since $q' \not> q$, we see that the q-element certainly exhibits at once by its equation its highest component. Thus the q-element contains a q-point with certain latent multiple points, no one of order $> q$. If these latent multiple points contain Q dps, we have therefore

$$\delta = q(q-1)/2 + Q.$$

Further, it was shown that for a single superlinearity of order k,

$$\varkappa = k - 1.$$

Regarding the reciprocal equation of degree r in the same way, we have

$$\lambda = r(r-1)/2 + R,$$

and for a single superlinearity of class l,

$$\iota = l - 1.$$

We have now to show how this same method of quadric inversion may be applied to determine λ and ι, and hence τ, without forming the reciprocal equa-

tion. The expressions just given for δ and λ are for any singularity, whether composite or simple. We first show how q and r present themselves; we then prove that in every case $R = Q$.

12. It has already been pointed out that any singularity is caused by the superposition of a number of superlinearities. Now if these come at the same point, with different tangents, no latent dps and no latent dls are caused; and similarly for the same tangent with different points of contact. Thus for our present purpose we have only to consider a q-element with all the tangents coincident.

A superlinearity at O, tangent OZ, when inverted gives rise to a superlinearity at Z; let OZ, the *invertor*, meet this in l points, and let the base meet it in k points, so that k is the order of the singularity O. The tracing point P (describing Z) (Fig. 6a) has therefore to cut the base k times and the

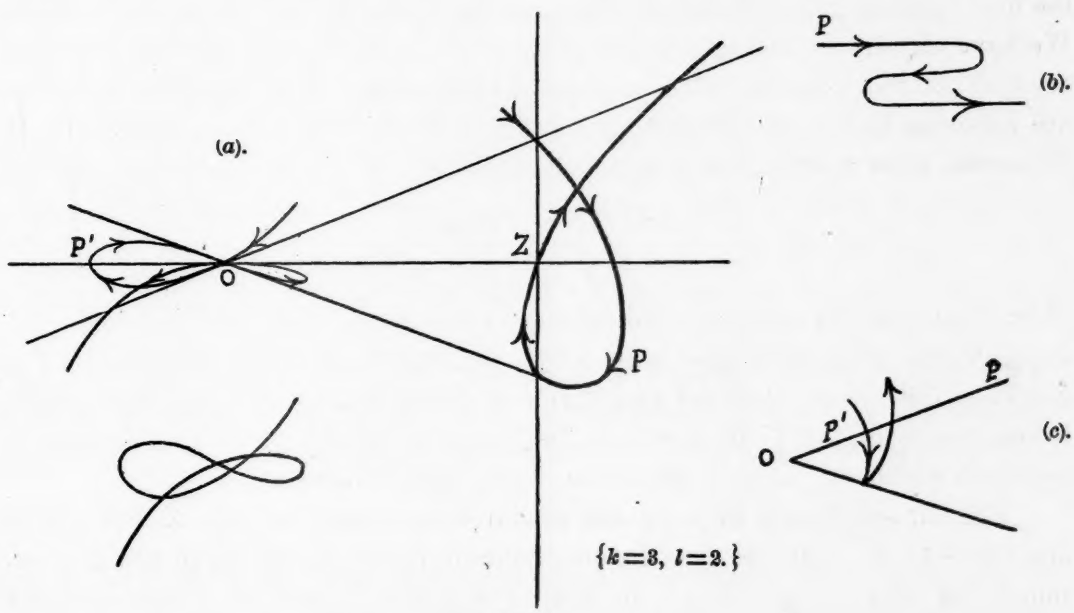

FIG. 6.

invertor l times. Now in order that it may cut the base k times, it must oscillate along OZ, making $k - 1$ turnings back (Fig. 6b); any such turning back is represented by consecutive points (which may be also coincident) on the base, and thus it produces a coincidence of consecutive points at O, i. e. a cusp; thus

the number of cusps is $k-1$, and k is the order of the singularity O, which agrees with what we have already said as to the degree of the point equation.

Again, in order that P may cut the invertor l times, the line OP, whose motion represents that of the tangent at O (viz. OP'), must take up the position OZ l times, and consequently attain a limiting position and turn back $l-1$ times (Fig. 6c). Now it is only at these turnings back that two consecutive positions of the tangent OP' can coincide, i. e. these $l-1$ are the only inflexional tangents that can present themselves. Thus OZ presents itself as an l-line, giving $l(l-1)/2$ dls, of which $l-1$ are inflexional tangents; the class of the original singularity is therefore l, and λ is at least $l(l-1)/2$. Thus the order and class of the superlinearity at O are given by the number of points in which the first inverse is met by the base and the invertor.

Now consider the singularity made by the superposition of j superlinearities with the same tangent OZ at O. Let these be of order and class $k_1 l_1$, $k_2 l_2$, etc. Then the first inverses, all at Z, are met by the base and invertor in Σk and Σl points. We have already defined the order of the singularity as Σk, the degree of the original point equation; and similarly the class is Σl; thus the order and class are given as before by the number of points at Z on the base and invertor. Moreover, since $\varkappa_1 = k_1 - 1$, $\iota_1 = l_1 - 1$, we have

$$\varkappa = \Sigma(k_1 - 1) = k - j,$$
$$\iota = \Sigma(l_1 - 1) = l - j.$$

Note then that OZ meeting the inverse in l points, and the base meeting it in k points, the number of points in which OZ meets the original singularity is $k + l$. Thus the order of the singularity at O is given by the degree, k, of the lowest terms in the point equation, and subtracting this from the number of points in which the tangent meets the singularity, we have the class, l.

For our singularity we have now as minimum values for δ and λ, $k(k-1)/2$ and $l(l-1)/2$. Let the first inverse contain, in all, h dps, these being determined as already explained, so that $\delta = k(k-1)/2 + h$; write similarly $\lambda = l(l-1)/2 + h'$; we have now to show that in every case $h' = h$.

13. The determination of the number of dls really depends on our knowledge of the general point nature of a singularity. We have a k-pt, from which proceed a certain number of strings of multiple points, in general of decreasing

order. Thus the singularity illustrated in Figs. 26, 27 (loc. cit.) gives a sextic point; the dps 4, 5, 6 indicate three series of dps; in two of these we have only the one member, 5, 6, but in the remaining one we have a series of four consecutive dps. Some of these series may be closed altogether or in part, the branches uniting to form a cusp; and finally we have emerging $2j$ branches, 2 for each superlinearity in the composite singularity. If now N is a k'-pt consecutive to O, k' of the k tangents at O pass through N; in the example already chosen the six tangents at O are coincident in pairs, leading to 4, 5 and 6; we have thus 3 dls. But also there are the emerging free arcs, which have contact of a certain order, and therefore yield a certain number of dls. So we have to take into account the dls that lead from O to the consecutive multiple points, those from the first rank of latent multiple points to the second, and so on.

Now two free branches that have v consecutive points common have in general v double tangents; but if the branches have inflexions at O, this number is increased. Let the two branches, each of order 1, be of class l_1, l_2. If we have two superlinearities of orders k_1, k_2, each of class 1, these lead on inversion to linear branches having k_1 and k_2 points on the base; thus the number of dps in the inverse is the smaller of the two numbers k_1, k_2; let this be k_1; the total number of dps is therefore $(k_1+k_2)(k_1+k_2-1)/2+k_1$, of which k_1-1+k_2-1 are cusps. Reciprocally, for the number of dls in the case to be considered, if $l_1 \not> l_2$, we have

$$\lambda=(l_1+l_2)(l_1+l_2-1)/2+l_1.$$

Now the singularity formed by the two branches is of order 2, class l $(=l_1+l_2)$; and if $l_1<l_2$, l_1 is the number of latent dps, for the two branches have respectively l_1+1 and l_2+1 points on the common tangent; thus

$$\lambda=l(l-1)/2+h;$$

but if $l_1=l_2$, there are other latent dps, say v_1 in one direction from O, v_2 in the other; but these, indicating a certain degree of contact between the curved branches, give respectively v_1 and v_2 dts, so that we have

$$\lambda=l(l-1)/2+l+v_1+v_2,$$

i. e. as before $$\lambda=l(l-1)/2+h.$$

If now we have any number of such branches, since the expression for λ brings in the dls for the two branches separately, this part must not be repeated in adding up the numbers found for $\lambda_{12}\,\lambda_{13}\,\lambda_{23}$, etc. We have

$$\begin{aligned} 2\lambda_{12} &= (l_1+l_2)(l_1+l_2-1)+2h_{12} \\ &= l_1(l_1-1)+l_2(l_2-1)+2l_1l_2+2h_{12}, \\ \therefore\ 2\Sigma\lambda_{12} &= \Sigma l_1(l_1-1)+2\Sigma l_1l_2+2\Sigma h_{12} \\ &= \Sigma l_1^2+2\Sigma l_1l_2-\Sigma l_1+2\Sigma h_{12} \\ &= \{\Sigma l_1\}^2-\Sigma l_1+2\Sigma h_{12} \\ &= l^2-l+2h, \end{aligned}$$

and so we obtain finally $\lambda = l(l-1)/2+h$.

Now the diagram for the penultimate form of a j-fold singularity of order k contains $k-j$ evanescent loops; if we remove the tips of f of these loops, so making the corresponding cusps into nodes, we obtain a diagram for the penultimate form of a $(j+f)$-fold singularity containing $k-(j+f)$ cusps; thus k and δ, and therefore h, are unaltered, but l and ι, and therefore λ, are in general altered. It is easiest to trace the changes when we start with the free branches, and restore the tips, i. e. reintroduce the cusps, one at a time. We have then to show that if $\lambda = l(l-1)/2+h$,

$$\lambda' = l'(l'-1)/2+h;$$

i. e. we have to show

$$\lambda'-\lambda = \tfrac{1}{2}\{l'(l'-1)-l(l-1)\}.$$

Now we shall find that in every case

$$\begin{aligned} & l'-l = 1 \text{ or } 0\text{; if then} \\ & l'-l = 0, \text{ we have to show } \lambda'-\lambda = 0 \\ \text{and if }\ & l'-l = 1, \text{ we have to show } \lambda'-\lambda = l. \end{aligned}$$

The tangents at a cusp are (i) consecutive, (ii) coincident; if consecutive, the arcs in the vanishing angle turn their concavities towards one another; if coincident, the branches are on (a) the same, (b) opposite sides. Further, (iii) removing a cusp may remove intersections with other branches, so this must be considered in restoring the cusp. (Fig. 7.)

In (i) l and λ are unaltered;

in (ii, a), l is unaltered; a double tangent is changed into an inflexional tangent, but this leaves λ unaltered;

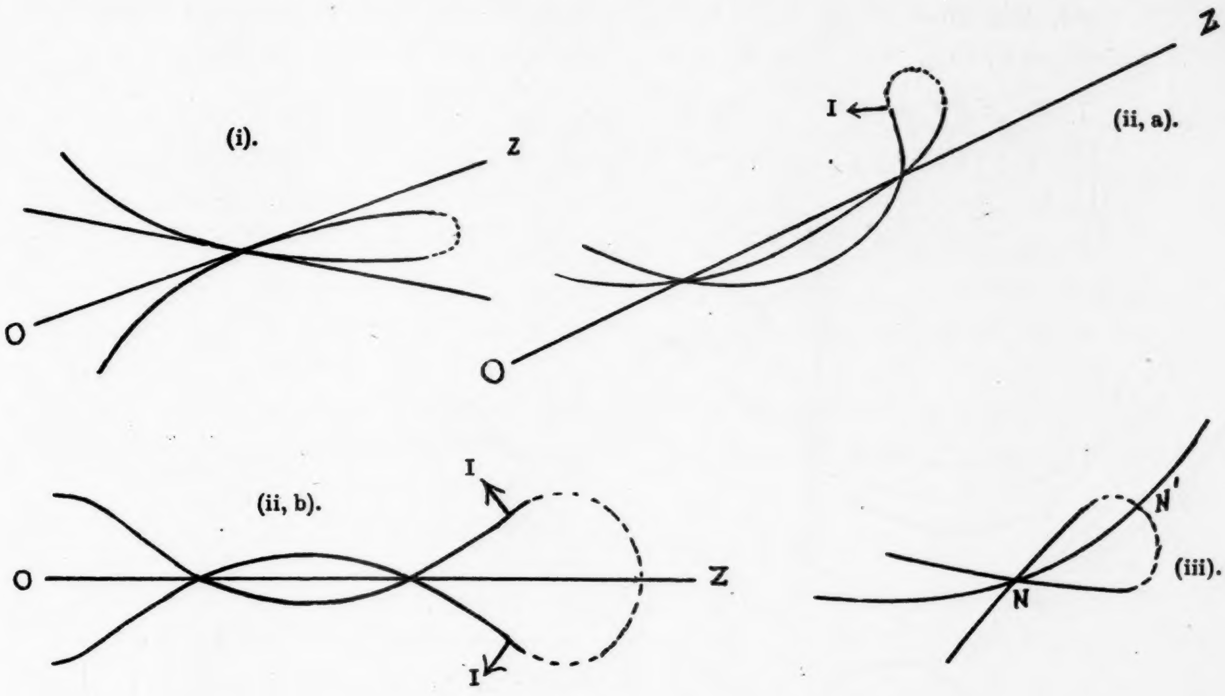

Fig. 7.

in (ii, b), l is increased by 1, ι by 2, without any corresponding diminution in τ, but with an actual increase. For OZ, which was an l-line, becomes an $(l+1)$-line; thus λ is increased in all by $\frac{1}{2}\{(l+1)\,l - l\,(l-1)\}$ i. e. by l;

in (iii), the joining of the free ends introduces one more dp N'; it does not additionally affect l, ι, but as NN' is now a dt, λ is increased by unity, corresponding to the increase of h by unity.

Thus throughout the changes we have

$$\lambda = l\,(l-1)/2 + h;$$

i. e. in any singularity with all the tangents coincident, *the number of latent dps is equal to the number of latent dls*; and as the superposition of singularities with different tangents introduces no latent double elements, this relation is universally true. We have then for a j-fold singularity the equations

$$\delta = k\,(k-1)/2 + h; \quad \lambda = l\,(l-1)/2 + h;$$
$$\varkappa = k - j; \qquad \iota = l - j.$$

14. Examples of the four kinds of superlinearity, enumerated in §10, with the values of (k, l, h). (Fig. 8.)

I. $(y - x^2)^3 = x^2y^3$; $(3, 3, 4)$; $\therefore \varkappa = 2, \nu = 5$; $\iota = 2, \tau = 5$.
II. $(y - x^3)^3 = x^2y^3$; $(3, 6, 7)$; $\therefore \varkappa = 2, \nu = 8$; $\iota = 5, \tau = 17$.
III. $(y^2 - x^3)^3 = x^4y^4$; $(6, 3, 7)$; $\therefore \varkappa = 5, \nu = 17$; $\iota = 2, \tau = 8$.
IV. $(y^2 - x^3)^2 = x^7 + 4x^5y$; $(4, 2, 2)$; $\therefore \varkappa = 3, \nu = 5$; $\iota = 1, \tau = 2$.

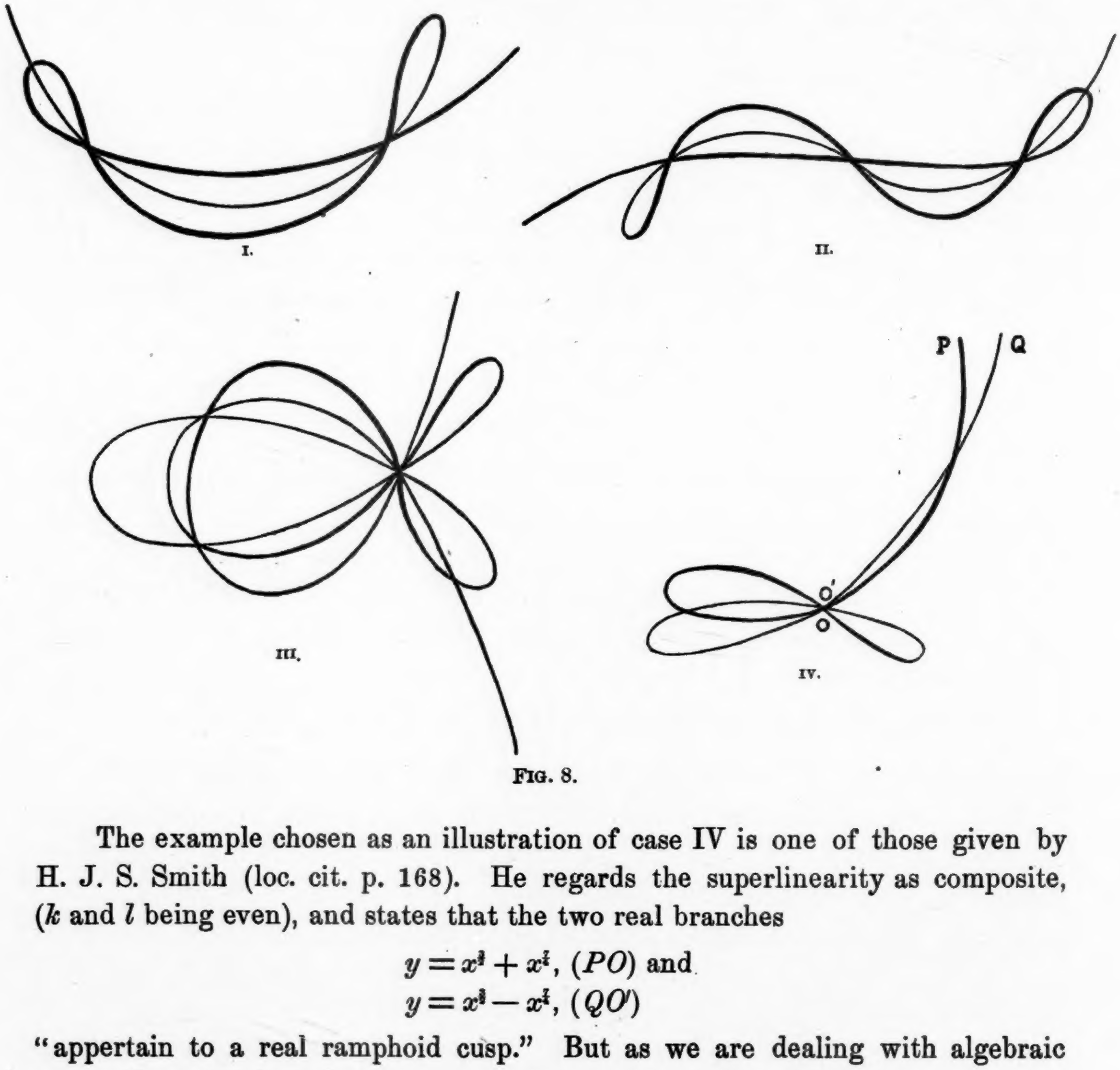

FIG. 8.

The example chosen as an illustration of case IV is one of those given by H. J. S. Smith (loc. cit. p. 168). He regards the superlinearity as composite, (k and l being even), and states that the two real branches

$$y = x^{\frac{3}{2}} + x^{\frac{7}{4}}, \ (PO) \text{ and}$$
$$y = x^{\frac{3}{2}} - x^{\frac{7}{4}}, \ (QO')$$

"appertain to a real ramphoid cusp." But as we are dealing with algebraic curves, these two branches have no existence without their conjugates; geometri-

cally, these are needed for the transition from O to O'. We therefore cannot regard the superlinearity as composite.

15. This quantity h, which has numerical analogies with the deficiency of a curve, may be conveniently called the *excess* of the singularity; and the relation proved in §13 shows that for any singularity the point excess and the line excess are equal. We have then a dualistic organic classification of superlinearities, by order, class, and excess, $(k, l;\ h)$ corresponding to the $(m, n;\ p)$ classification of curves; and just as in the case of curves, these characteristic numbers do not lead to a unique determination; for this, in the case of even a single superlinearity, we require to know the series of multiple points and their arrangement. The Plückerian characteristics are expressed in terms of $k\ l\ h$ by the equations given in §13, from which we obtain

$$\nu = (k-1)(k-2)/2 + h, \quad \tau = (l-1)(l-2)/2 + h;$$

and we can derive a series of equations analogous to those given in Salmon's Higher Plane Curves, p. 65; e. g.

$$\begin{aligned} k^2 - 2\nu - 3\varkappa &= l^2 - 2\tau - 3\iota; \\ 2(\tau - \nu) &= (l-k)(l+k-3) \\ &= (\iota - \varkappa)(\iota + \varkappa - 1) \text{ (cf. with H. J. S. Smith, p. 166)}; \\ \nu - (k-1)(k-2)/2 &= \tau - (l-1)(l-2)/2 = h. \end{aligned}$$

Further, as to the analogy of the excess h with the deficiency p; when the singularity is resolved into a series of multiple points, the maximum number of such multiple points is $h+1$; for the maximum is plainly attained if all the latent multiple points are dps, and then there must be h of them, so that counting in the central k-pt we have $h+1$ multiple points. And again, we can determine a minimum value for h. Let the smaller of the two numbers $k, l = g$; if $g > 1$, the point Z on the first inverse is a multiple point of order g, and therefore contains $g(g-1)/2$ dps; thus $h \nless g(g-1)/2$.

16. In certain cases when k and l are given, h is known. For we start with a singularity O, of order k and class l, or say a point $(k,\ l+k)$. Let the quotients and remainders in the process of finding the G. C. M. of $k,\ l+k$ be $f, f_1, f_2 \ldots . k_1, k_2 \ldots .$ so that

$$l + k = fk + k_1, \quad k = f_1 k_1 + k_2, \text{ etc.}$$

Then f inversions along OZ lead to a point (k_1, k); f_1 inversions along the new tangent (which was the base) lead to a point (k_2, k_1), and so on. (Fig. 9.) This continues until we come to a point for which $k_r = k_{r+1}$, and then the tangent is neither of the lines with which we are dealing, so that we do not know in how

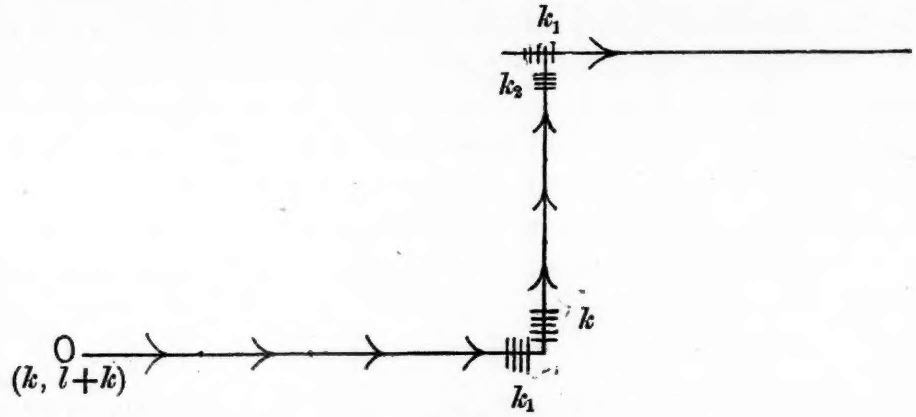

FIG. 9.

many points the tangent meets the singularity, and therefore cannot say à priori how the series of multiple points continues. But if k is prime to l, the last remainder is unity, and so the series of multiple points terminates. We have then f k-pts, f_1 k_1-pts, ... f_ν k_ν-pts; consequently we have

$$\begin{aligned}
2\delta &= f \times k(k-1) + f_1 \times k_1(k_1-1) + \dots + f_\nu \times k_\nu(k_\nu - 1) \\
&= (l+k-k_1)(k-1) + (k-k_2)(k_1-1) + (k_1-k_3)(k_2-1) + \dots \\
&\qquad \dots + (k_{\nu-1}-1)(k_\nu-1) \\
&= (l+k)(k-1) - kk_1 + kk_1 - k_1k_2 + k_1k_2 \dots + k_{\nu-1}k_\nu \\
&\qquad + k_1 - k + k_2 - k_1 + \dots + k_\nu - k_{\nu-1} - k_\nu + 1 \\
&= (l+k)(k-1) - k + 1 \\
&= (k-1)(l+k-1);
\end{aligned}$$

now $2\delta = k(k-1) + 2h$,

$$\begin{aligned}
\therefore \quad 2h &= (k-1)(l+k-1) - k(k-1) \\
&= (k-1)(l-1);
\end{aligned}$$

i. e. if l is prime to k, $h = (k-1)(l-1)/2$. (Cf. w. Brill, loc. cit., p. 355.)

17. The considerations already presented as to the arrangement of the constituent multiple points in a superlinearity show that there is a twisting

backwards and forwards on a certain fundamental curve, with, it may be, at O a "rosette" as in the point referred to in §13. In considering possible arrangements, it should be noted that if in a series of multiple points the order drops by unity, we have got to the end of that series. For the point before this one being $(k', l' + k')$, certainly k' is not unity, or we should have got to the end; now the next point is (k', l'), and we have therefore by the hypothesis $l' = k' - 1$; i. e. we have now got to a point of order $k''(= l')$ met by its tangent in $k'' + 1 (= k')$ points; the inverse of this is then of order $k'' + 1 - k''$, i. e. 1; e. g. $y^3 = x^{11}$; here we have 3pt + 3pt + 3pt + 2pt; there are consequently no more multiple points. The converse of course does not hold.

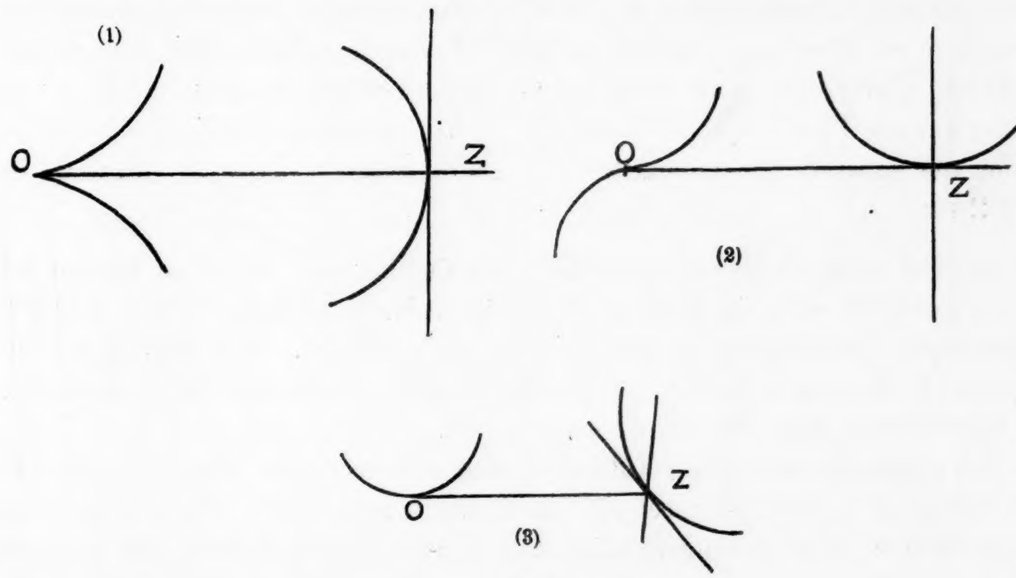

Fig. 10.

18. Since every superlinear branch can be reduced by successive inversions to an ordinary branch, we might regard any one as so many degrees removed from an ordinary arc, or say as of such a rank, counting the ordinary arc as of the first rank. Now a singularity at Z yields different results at O according as (1) the base, (2) the invertor, or (3) some other line, is the tangent at Z. (Fig. 10.) But if the arc at Z is a one-element, then inverting by process (3) gives no higher singularity at O; rejecting this, in the first rank we have one member; in the second we have *two*. (Fig. 10; (1) (2).) In the third we have

$3 \times 2 - 1$, since the inverse of (2) by the third process is to be rejected; and in general in rank g there are $\frac{1}{2}\{3^{g-1}+1\}$ members. But it hardly seems advisable to adopt this classification, for if we attempt to enumerate singularities in this way we shall be taking account of insignificant differences. Thus, e. g. $(y-x^2)^2 = x^7$ and $(y-x^2-x^3)^2 = x^7$ would be enumerated as distinct; now each has one cusp and two nodes, arranged as consecutive points on a curve of finite curvature; each has one inflexion and two double tangents; the only difference is that the foundation curve is in one $y = x^2$, and in the other, $y = x^2 + x^3$.

19. Since a composite singularity may be made by any combination of superlinearities, it hardly seems worth while attempting an organic classification. For even if we introduce j for the number of superlinearities, (§13) though the Plückerian characteristics are thus made to depend on the essential nature of the singularity, yet the four quantities k, l, h, j, do not even tell what the numbers are for the separate superlinearities; so there seems no special advantage in using them.

20. The relations proved show that two singularities related by having the same inverse, but with the base and invertor interchanged, have their l and k interchanged, and are thus reciprocal as to their symbol. Are they in reality reciprocal? We know that $(k, l; h)$ do not uniquely determine the singularity, and thus there is room for asking this question.

Take a proper conic, with $OZ\Omega$ a self-conjugate triangle. (Fig. 11.) Let QP' be the polar of P, then OP meets QP' in P', the inverse of P. Let Q be the pole of OP; then if OP is the tangent at O, and therefore coincides with OZ, Q comes at Ω, and $\Omega P'$ is the tangent to the reciprocal. Thus the singularity at Z, when inverted with ΩZ as the invertor, gives at Ω a singularity which, to a certain order of approximation, coincides with the reciprocal of O. As to the branches proceeding from the k-pt, we know that these reciprocate into branches having contact of the same order. Thus the central part of the singularities at O and Ω will be reciprocal; and the number and general arrangement of the outlying multiple (reciprocal) elements will correspond; but the two singularities will not necessarily be reciprocal.

21. It is of interest to compare the formulæ here obtained with those given by Halphen in his "Mémoire sur les points singuliers des courbes algébriques

planes" (1874) for the behavior of the evolute. The generalised evolute is the envelope of the line $O\Omega$ in Fig. 11; this is the polar of Z. For parts of the curve in the immediate neighborhood of O, with OZ as tangent, the generalised evolute and the first inverse are reciprocal polars. If we take the base conic a circle, so that IJ are the circular points, the generalised evolute coincides with the ordinary evolute for parts of the curve in the immediate neighborhood of O;

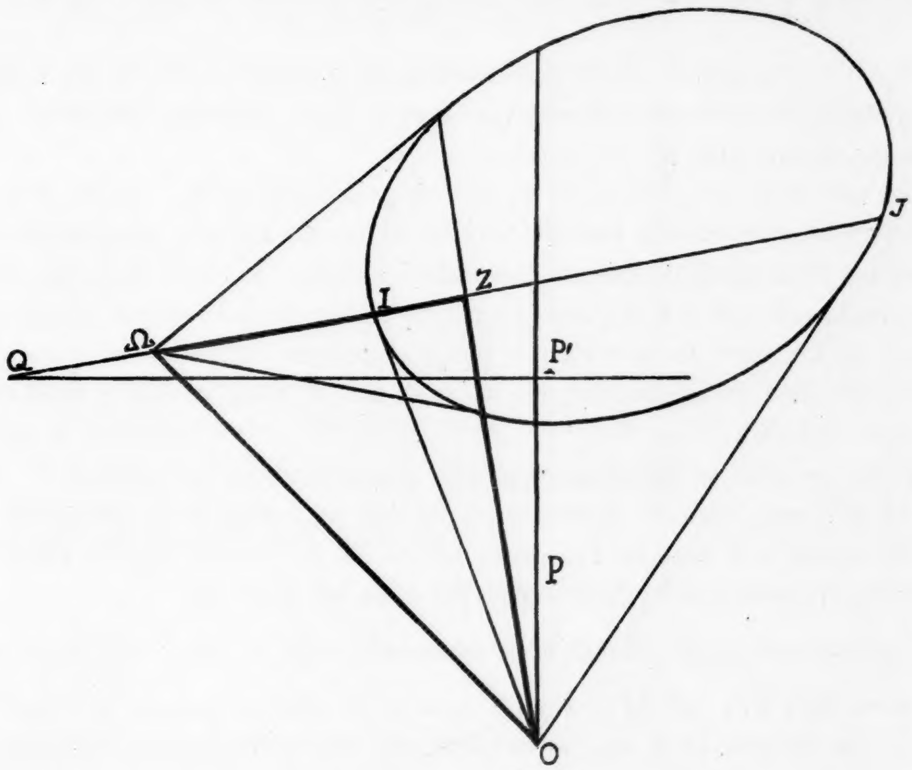

FIG. 11.

and thus for these parts of the curve the first inverse coincides with the reciprocal polar of the evolute. Now Halphen's formulæ, expressed in terms of k, l, show that the order of the singularity on the evolute is $k \sim l$, and if $k < l$ this singularity is at infinity, and the tangent meets it in l points; if $k > l$, it is at O, and the tangent meets it in k points. As to the first inverse, our results show that if $k < l$ the order of the singularity is k, and the tangent meets it in l points; while if $k > l$ these numbers are interchanged. Thus the order and

class for the singularity on the evolute are in the two cases $(l-k, k)$, $(k-l, l)$; and for the singularity on the inverse they are $(k, l-k)$, $(l, k-l)$. Further, if $k < l$, OZ is the tangent at Z, and the reciprocal is at Ω; if $k > l$, IJ is the tangent at Z, and the reciprocal is at O; and if $k = l$ (a case in which a priori formulæ do not present themselves, and which Halphen therefore does not consider), the tangent is a line through Z inclined at a finite angle to OZ, ΩZ, and the reciprocal is somewhere on $O\Omega$, i. e. it is at a finite distance from O.

22. We have now to show that the reduction in class due to the compound singularity is the same as the reduction due to the constituent elements (nodes and cusps) enumerated by this process.

Let the total diminution due to the singularity of order k be D; when this singularity is inverted, we have a part of it preserved as a singularity on the inverse—let the effect of this on the class be D_1, so that $D - D_1$ is the effect of the central k-point in the original singularity; this central k-point is replaced by k points on the base, these giving $\varkappa_1$ proper contacts (improper contacts, that is cusps in the inverse singularity, are allowed for in D_1); we have to show that $D - D_1 = k(k-1) + \varkappa_1$.

Take the conic of inversion a proper conic, choosing it so that I, J, as also OI, OJ, IJ, may have no specialty of position with regard to the given curve. Let the order and class of the given curve be m, n; and let the effect of all remaining singularities in diminishing the class be N, so that

$$n = m(m-1) - D - N. \tag{1}$$

The curve cuts OI, OJ, IJ in $m-k$, $m-k$, m, distinct points; we have therefore on the inverse, at I, an $(m-k)$-pt with distinct tangents; the same at J; and at O an m-pt with distinct tangents. On IJ we have also k points, and consequently

$$m' = 2(m-k) + k,$$

$$\text{i. e.}\quad m' = 2m - k; \tag{2}$$

hence

$$n' = m'(m'-1) - D_1 - N - m(m-1) - 2(m-k)(m-k-1), \tag{3}$$

for the ordinary proof, which is legitimate when the tangents are separate, shows that the diminution in class due to an m-pt is $m(m-1)$.

Now the n' tangents from I to the inverse are

(1) the inverses of the original n J-tangents;

(2) $m - k$ tangents at I, each counting twice;

(3) the line IJ, counting $\varkappa_1$ times;

$$\therefore\ n' = n + 2(m-k) + \varkappa_1. \tag{4}$$

Comparing (3) with (4), and making use of (1) and (2), we have

$$\begin{aligned}(2m-k)(2m-k-1) - D_1 - N - m(m-1) - 2(m-k)(m-k-1)& \\ = m(m-1) + 2(m-k) + \varkappa_1 - D - N&,\end{aligned}$$

i. e.

$$\begin{aligned}D - D_1 = \varkappa_1 + 2(m-k) + 2(m-k)(m-k-1)& \\ -(2m-k)(2m-k-1) + 2m(m-1)&,\end{aligned}$$

i. e.

$$D - D_1 = k^2 - k + \varkappa_1 = k(k-1) + \varkappa_1. \tag{5}$$

Now the central k-point gives in all $k(k-1)/2$ dps, of which we have supposed $\varkappa_1$ to be cusps; we have therefore $\nu_1 = k(k-1)/2 - \varkappa_1$, and equation (5) becomes

$$D - D_1 = 2\nu_1 + 3\varkappa_1;$$

that is, the ν_1 nodes and $\varkappa_1$ cusps in composition have the same effect in reducing the class as when separate. We have therefore the equation

$$n = m(m-1) - 2\Sigma\nu - 3\Sigma\varkappa;$$

and from the reciprocal curve we have

$$m = n(n-1) - 2\Sigma\tau - 3\Sigma\iota;$$

where the $\Sigma\varkappa$, $\Sigma\iota$, refer to all consecutions, the $\Sigma\nu$, $\Sigma\tau$, to all remaining coincidences.

BRYN MAWR COLLEGE, BRYN MAWR, PA., *March*, 1893.

The Elliptic Inequalities in the Lunar Theory.

By Ernest W. Brown, *Fellow of Christ's College, Cambridge.*

The object of this paper is to give a general solution in series of the problem "A system of three bodies is in motion in one plane, the first is revolving about the second and is disturbed from its elliptic orbit by the third. The third body is supposed to be of infinite mass and to be moving in a circle of infinite radius with a finite angular velocity. Given the relative positions of the three bodies at any one time, to find their relative positions at any other time." The differential equations which express the motion of the first body relative to the second have been given by Dr. Hill in Vol. I of this journal. The second body is taken as the origin, and the motion is referred to rectangular axes moving with the angular velocity of the third body which always lies on the x-axis. The equations are

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right)x = 0,$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}y = 0,$$

where μ is the sum of the masses of the first and second bodies and n' the angular velocity of the third about the origin.

One integral of these equations is immediately obtained. It is

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{r} + 3n'^2x^2 - 2C.$$

In chapter II of his memoir just referred to, Dr. Hill transforms these equations to the imaginary variables $x \pm y\sqrt{-1}$, and by the use of this, the Jacobian Integral, brings them to two simultaneous differential equations of the second order and second degree.

A particular solution is then found which can be expressed by an infinite trigonometrical series with a single period, giving as an orbit relative to the moving axes, a closed symmetrical curve which, for small values of the ratio of the mean motions of the first and third body, is an oval. I have given the extension of this particular solution to the case where the third body is at a comparatively great but not infinite distance.* The orbit is still a closed curve, but it has lost its symmetry about the axis of x. This extension presented no special difficulties beyond the question of representing the series obtained in the most convergent form.

When an attempt is made to obtain the general solution, it appears below that it will be necessary to represent it by an infinite series with two periods; that is, by a doubly infinite trigonometrical series. The orbit is now no longer a closed curve, but it is one which re-enters its orbit after an infinite time, and the maxima and minima values of the coordinates lie between determinate limits. The chief difficulty lies in the fact that the equations of condition between the coefficients of the terms in the series require a relation between the two periods; it is the finding of this relation that entails the trouble.

Again, since the solution is expressed by an infinite series, it is necessary in order that it should have a meaning, that the series should be convergent, for this, two of the arbitrary constants must be confined within certain limits. We are thereby limited to consider only disturbing bodies whose mean motions have ratios to that of the revolving body which do not surpass a certain value, and also that the body when undisturbed shall have an eccentricity less than some given quantity. It does not seem easy to determine what these limits are or whether they exist at all. M. Poincaré's remarkable researches† into the methods of the problem of three bodies have to a large extent opened the way to the determination of convergency. He has in fact proved that for small values of certain of the constants, there are periodic solutions which can be represented by convergent series. The convergence of the series obtained below will depend in a large measure on the value of c, the ratio of the two periods of the general solution.

The principal part of the ratio c has been shown to be one of the roots of an equation expressed as an infinite determinant, and it was this peculiarity that

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* American Journal of Mathematics, Vol. XIV, p. 141.

† Acta Mathematica, Vol. XIII. Mécanique Celeste, Vol. I.

stimulated inquiry into the possibility of the existence of such forms. G. W. Hill, Poincaré and Helge von Koch have given investigations of their properties. In particular, Dr. Hill,* under the assumption that it was possible, actually obtained the principal part of c from the infinite determinant which arose in his method; later M. Poincaré† gave conditions which, when satisfied, allowed the determinant to possess a finite and determinate value, and also proved that it was possible to obtain c from it; and lastly, Dr. von Koch‡ has shown under what circumstances the ordinary transformations used in determinants with a finite number of rows and columns can be extended when these latter are infinite in number.

If we examine the series given by Delaunay in longitude and parallax for the coefficients of the several trigonometrical terms, it appears that at any rate in the case of the moon, slow convergence, whenever it occurs, proceeds in any coefficient along the series arranged in powers of the ratio of the mean motions; also that in terms with actually small coefficients the convergence appears to be slower than in the larger coefficients. As pointed out by Dr. Hill, it becomes of interest to see how these can be improved either by taking new quantities in terms of which to make the expansions or by using numerical values from the start. He has examined this question in the case of the principal part of the Variational Inequalities, and I have attempted to give convenient expressions for the same purpose in that of the Parallactic Inequalities. When the equations for the Elliptic Inequalities are examined, it does not appear that there is any method of improving the expressions for the coefficients on the same lines beyond taking $m = n'/(n - n')$ instead of n'/n in terms of which to make expansion. The quantity c, which is itself represented by a slowly converging series in powers of m, is involved in a complex manner, and even if it were not, we should simply be brought back to the difficulty of representing c by a quickly convergent series. Hence it seems better to use the numerical value of m from the outset, *keeping all the other constants involved arbitrary*; that is to say, in the case treated here the eccentricity is left arbitrary.

Another consideration justifies this method of proceeding. Of all the constants used in the determination of the motion of the moon, the ratio of the mean motions is most exactly determined, and it is doubtful whether any future observations will change it to any appreciable extent. This is not the case with

* Acta Math., Vol. VIII.

† Bulletin de la Soc. Math. de France, Vol. XIV, p. 77.

‡ Acta Math., Vol. XVI, p. 217.

the other constants. As the slow convergence occurs in the series arranged in powers of this ratio, little will be lost by keeping the rest of the constants arbitrary, and the extra trouble on the plan here adopted is very small. If then a complete theory of the moon were given on this principle, the coefficient of every term would appear as a quickly convergent series arranged in powers of the eccentricities, the ratio of the parallaxes, and the sine of the latitude, with numerical multipliers only. The coefficients in such a development would have a comparatively small number of terms with a high degree of accuracy.

Another point may also be noted in connection with these methods. It appears generally that in proceeding to find the coefficients by successive approximation, the second and higher approximations are due principally, not to the introduction of new terms in the series, but to the second and higher approximations of the terms previously used, so that it is not generally necessary to find the first approximations of very many of the higher terms in order to get the large coefficients accurately. This fact abbreviates the calculations to a large extent.

With respect to the numerical results obtained, no trouble has been spared to make them accurate. Equations of verification have been computed for each step and also for the final results. The various forms which the equations of motion can take enable this to be easily done. In the second portion of the paper (to be published hereafter) a value has been obtained for the part of the motion of the Lunar Perigee which depends on the square of the eccentricity. In order that this value should possess a degree of accuracy which can be relied on, it is absolutely necessary that the quantities obtained be verified by independent equations. I have not had the advantage of any assistance in carrying out the computations, and have consequently taken special pains to find out the best equations for verification, and have computed in certain cases two and even three of them to test the same results.

I.

The equations under discussion are

$$\frac{d^2x}{dt^2} - 2n' \frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right) x = 0,$$

$$\frac{d^2y}{dt^2} + 2n' \frac{dx}{dt} + \frac{\mu}{r^3} \cdot y \qquad = 0,$$

which admit of a general solution when n' is zero and a particular solution when n' is not zero. In the first case the resulting orbit is an ellipse, and therefore, as is well known, x, y can be expressed in terms of the time by means of Bessel's functions*

$$x = a\sum_{-\infty}^{+\infty} {}_p \frac{1}{p} J_{\frac{pe}{2}}^{(p-1)} \cos pg\,(t - t_0),$$

$$y = b\sum_{-\infty}^{+\infty} {}_p \frac{1}{p} J_{\frac{pe}{2}}^{(p-1)} \sin pg\,(t - t_0),$$

where $\frac{1}{0} J_0^{(-1)} = -\frac{3}{2}e$, $b^2 = a^2(1 - e^2)$ and g is the mean motion. The four quantities, a, e, g, t_0, involve three arbitrary constants, the fourth being used in making the major axis of the ellipse coincide with the x-axis. The particular solution of the equations involves only two arbitrary constants, and is of the form†

$$x = a_0 \sum_{-\infty}^{+\infty} {}_i\, a_i \cos(2i + 1)(n - n')(t - t_1),$$

$$y = a_0 \sum_{-\infty}^{+\infty} {}_i\, a_i \sin(2i + 1)(n - n')(t - t_1),$$

the arbitraries being n (or a_0) and t_1 (when $i = 0$, $a_i = 1$). These two forms are distinct unless both e and n' are zero, when they reduce to circular motion. The coefficients a_i are functions of the ratio $n' : n - n'$. Since then we have these forms for zero values of n' and e respectively, *it is assumed* that for small values of e and n' there is a possible solution of the form

$$\left.\begin{aligned} x &= a_0\Sigma_i\Sigma_p A_{i,\,p} \cos\{(2i+1)(n-n')(t-t_1) + pg\,(t-t_0)\}, \\ y &= a_0\Sigma_i\Sigma_p A_{i,\,p} \sin\{(2i+1)(n-n')(t-t_1) + pg\,(t-t_0)\}, \end{aligned}\right\} (i, p = +\infty \ldots . -\infty).$$

This solution involves the necessary four arbitrary constants and may be looked upon as a general solution of the equations except in so far as e and $n'/(n - n')$ are limited in magnitude. This limitation is necessary in order that the series may be convergent, but it is to be understood that there is no guarantee that the

* Tisserand, Méc. Cel. t. I, p. 226.

† G. W. Hill, Researches in the Lunar Theory, Amer. Jour. Math., Vol. I, p. 130. Unless otherwise stated, the references to this memoir are given by the page of Vol. I of this journal.

series are convergent for even small values of these constants. At present all we know is that the solution with n' zero converges for values of e sufficiently small, and the particular solution for sufficiently small values of the ratio $n'/(n-n')$.

The relation $g : n - n'$ will involve some of the constants. Put then

$$g = c(n-n'),$$
$$\alpha = \{(2i+1)(t-t_1) + pc(t-t_0)\}.$$

Transform the equations of motion by putting $u = x + y\sqrt{-1}$, $s = x - y\sqrt{-1}$, and

$$-\frac{\sqrt{-1}}{n-n'} \cdot \frac{d}{dt} = D, \quad \frac{n'}{n-n'} = m;$$

they become, with the help of the Jacobian Integral (Vol. I, p. 132),

$$\left.\begin{aligned} D^2(us) - Du.Ds - 2m(uDs - sDu) + \frac{9}{4}m^2(u+s)^2 &= C, \\ D(uDs - sDu - 2mus) + \frac{3}{2}m^2(u^2 - s^2) &= 0. \end{aligned}\right\} \qquad (1)$$

We have
$$D(\alpha\sqrt{-1}) = 2i + 1 + cp,$$

and since the coefficients of the various powers of $e^{\alpha\sqrt{-1}}$ for all values of i, p (except the continuation $i = p = 0$) will be equated to zero, the constants t_0 and t_1 will not enter into the equations of condition thus found. They may be therefore omitted in what follows, care being taken that they be inserted when the values of the coefficients and of c have been found and the series written out in full.

Hence we can put

$$\alpha = (2i + 1 + cp)(n-n')\tau,$$

where $d\tau = dt$. Also write

$$e^{(n-n')\tau\sqrt{-1}} = \zeta,$$

so that

$$D = \zeta\frac{d}{d\zeta};$$

and the assumed solutions become

$$\left.\begin{aligned} u &= \Sigma_i\Sigma_p A_{i,\,p}\,\zeta^{2i+1+cp}, \\ s &= \Sigma_i\Sigma_p A_{-i-1,\,-p}\,\zeta^{2i+1+cp}, \end{aligned}\right\} \qquad (2)$$

Substitute these in the equations (1). For this purpose

$$
\begin{aligned}
Du &= a_0 \Sigma_i \Sigma_p (2i + 1 + cp) A_{i,\,p} \zeta^{2i+1+cp}, \\
Ds &= a_0 \Sigma_i \Sigma_p (2i + 1 + cp) A_{-i-1,\,-p} \zeta^{2i+1+cp}, \\
us &= a_0^2 \Sigma_i \Sigma_p \Sigma_j \Sigma_q A_{j,\,q} A_{-i+j,\,-p+q} \zeta^{2i+cp}, \\
&\text{etc.},
\end{aligned}
$$

where i, j, p, q take all integral values from $+\infty$ to $-\infty$. Equate to zero the coefficients of the various powers of ζ for all values of i, p, in order to obtain the requisite equations of condition between the coefficients. It is necessary that cp should not be an integer for any value of p; otherwise a set of equations of condition would be lost and an indeterminateness would result. As far as can be seen, cp is not an integer for any finite values of p. With this proviso there results

$$
\left.
\begin{aligned}
&\Sigma_j \Sigma_q [(2i+cp)^2 - (2j+1+cq)(2i+cp-2j-cq-1) - 2m(2i+cp-4j-2cq-2) \\
&\qquad + \tfrac{9}{2} m^2] A_{j,\,q} A_{j-i,\,q-p} + \tfrac{9}{4} m^2 \Sigma_j \Sigma_q (A_{j,\,q} A_{i-j-1,\,p-q} + A_{j,\,q} A_{-i-j-1,\,-p-q}) = 0, \\
&(2i+cp) \Sigma_j \Sigma_q [2i+cp-4j-2cq-2-2m] A_{j,\,q} A_{j-i,\,q-p} \\
&\qquad + {}^3 m^2 \Sigma_j \Sigma_q (A_{j,\,q} A_{i-j-1,\,p-q} - A_{j,\,q} A_{-i-j-1,\,-p-q}) = 0,
\end{aligned}
\right\} \quad (3)
$$

where in these equations j and q take all values from $+\infty$ to $-\infty$, and the equations are true for all values of i and p within the same range with a single exception—when $i = p = 0$, the right-hand side of the first equation is C, the constant of the Jacobian integral.

It will be noticed, either from the way in which these equations are obtained or immediately from their form, that the coefficients in square brackets are directly deducible from those of Hill's equations (Vol. I, p. 133) by putting in the latter $i + cp$ for i and $j + cq$ for j. The process he gives for isolating the terms of principal importance in corresponding pairs of equations can therefore be employed here.

Multiply the first equation by 2 and subtract the second multiplied by 3; we obtain

$$
\Sigma_j \Sigma_q [f(j, i, p, q) A_{j-i,\,q-p} A_{j,\,q} + 9m^2 A_{-i-j-1,\,-p-q} A_{j,\,q}] = 0,
$$

where

$$
\begin{aligned}
f(j, i, p, q) = &-(2i + cp)^2 + 4(2i + cp) + 2 + 4(2i + cp + 1)(2j + cq) \\
&+ 2(2j + cq)^2 + 2m\{2i + cp + 4(2j + cq) + 4\} + 9m^2.
\end{aligned}
$$

This single equation includes both equations by giving positive and negative values to i and p. The previous equations were sufficient if only positive *or* negative values were taken. A corresponding pair of equations is defined by the values i and p, $-i$ and $-p$. The terms of principal importance for obtaining $A_{i,p}$ and $A_{-i,-p}$ are got by putting in the first of this pair of equations $q=p, j=i$ and $q=0, j=0$, and in the second $q=0$, $j=0$ and $q=-p$, $j=-i$. These terms are respectively then

$$f(i, \quad i, \quad p, p)\,A_{i,p}A_{0,0}+f(\ \ 0, \quad i, \quad p, \quad 0)\,A_{-i,-p}A_{0,0},$$
$$f(0, -i, -p, \ 0)\,A_{i,p}A_{0,0}+f(-i, -i, -p, -p)\,A_{-i,-p}A_{0,0}.$$

Regard the equations as simultaneous ones to determine $A_{i,p}$ and $A_{-i,-p}$, solve so as to obtain $A_{i,p}$ and $A_{-i,-p}$ separately. The common divisor is

$$12\,(2i+cp)^2\left[2\,(2i+cp+1)(2i+cp-1)-4m+m^2\right],$$

and either equation is included in

$$\left.\begin{aligned}\Sigma_j\Sigma_q\,\{(j, i, p, q)\,A_{j,q}A_{j-i,q-p}+(i, p)\,A_{j,q}A_{i-j-1,p-q}&\\ +[i, p]\,A_{j,q}A_{-i-j-1,-p-q}\}&=0,\end{aligned}\right\}\ (4)$$

where

$$(j, i, p, q)=-\frac{\begin{aligned}(2j+cq)\{(2i+cp)^2-(2+4m-m^2)+(2i+cp)(2+2m)\}&\\ +(2j+cq)^2(2i+cp-2-2m)&\end{aligned}}{(2i+cp)\{2\,(2i+cp)^2-2-4m+m^2\}},$$

$$(i, p)=-\frac{3m^2}{4\,(2i+cp)^2}\,\frac{(2i+cp-2)(2i+cp-2-2m)-6-12m-9m^2}{2\,(2i+cp)^2-2-4m+m^2},$$

$$[i, p]=-\frac{3m^2}{4\,(2i+cp)^2}\,\frac{(2+10i+5cp)(2i+cp-2-2m)+6+12m+9m^2}{2\,(2i+cp)^2-2-4m+m^2}.$$

In this equation the multiplier of $A_{i,p}A_{0,0}$ or (i, i, p, p) is -1 and that of $A_{-i,-p}\,A_{0,0}$ or $(0, i, p, 0)$ is 0. By making the substitutions mentioned above, this equation might also have been deduced directly (Vol. I, p. 135). It is evident that all through the transformations cp is associated with $2i$ and cq with $2j$. The single equation (4) includes all the equations when i and p receive positive and negative values. When $i=p=0$, the right-hand side is no longer zero, but depends on the constant of Jacobi's integral and connects the latter with the other four constants introduced into the assumed solution.

Let us look for a moment at the way in which the primary forms are obtained from this general form. For the first form take the set of equations

given by $i = 0$ and p all integral values from $+\infty$ to $-\infty$, and put in them $m = 0$; since they reduce to the given forms for the first solution, this entails $j = 0$ only. In addition as $n' = 0$ and $g = cn$, n becomes equal to g and $c = 1$. The coefficients (i, p) and $[i, p]$ disappear, and we have for the determination of the Bessel's functions

$$\Sigma_q \frac{q}{p}\left[1 - \frac{(p-q)(p-2)}{2p^2-2}\right] A_{0,\,q} A_{0,\,q-p} = 0, \quad p = +\infty \ldots\ldots -\infty$$

or $$\Sigma_q q\left[p^2 + q(p-2) + 2p - 2\right] A_{0,\,q} A_{0,\,q-p} = 0,$$

(except when $p = 0$, when the right-hand side is C), and the assumed solution becomes, when the axis of the ellipse is made to coincide with the axis of x; that is, when $t_1 = t_0$,

$$x = a_0 \Sigma_p A_{0,\,p} \cos(p+1) g(t - t_0),$$
$$y = a_0 \Sigma_p A_{0,\,p} \sin(p+1) g(t - t_0).$$

Comparing, we obtain

$$a_0 A_{0,\,0} + a_0 A_{0,\,-2} = a,$$
$$a_0 A_{0,\,0} - a_0 A_{0,\,-2} = b,$$

and as a first approximation

$$A_{0,\,0} = 1, \quad A_{0,\,-1} = -\tfrac{3}{2} e, \quad A_{0,\,1} = e,$$

and $A_{0,\,p}$ is of the order e^p at least.

The particular solution is deduced by making $p = 0$ and $q = 0$ or $e = 0$, when it reduces to that given by Dr. Hill. Hence $a_0 . A_{i,\,0} = \mathrm{a}_i$, in his notation. As a_i is of the order m^{2i} at least, it is evident that unless the denominators contain powers of m as factors, $A_{i,\,p}$ is of the order $m^{2i} e^p$ at least.

The equations can then, provided m and e be sufficiently small, be solved by successive approximation, and this method is used below. In certain cases a power of m will appear in the denominator, but this does not affect the principle, as even then each approximation carries us at least two orders higher in m; in the solution given above it is impossible for more than the first power of m to be contained as a factor in the denominator.

II.

The system of denominators

$$(2i + cp)\left[2(2i + cp)^2 - 2 - 4m + m^2\right]$$

deserves special examination. The value of c is discussed below. For the pur-

pose in view here, I assume that it is known and that it has the expression in series given by Delaunay* or the numerical value given by Dr. Hill for its principal part.† These are

$$c = 1.071583\ldots,$$
$$c = 1 + m - \tfrac{3}{4}m^2 - \tfrac{201}{32}m^3\ldots, \text{ where } m = n'/(n - n').$$

Consider the algebraical expression of the two factors of this denominator-system when the second value of c has been substituted. The terms independent of m can only vanish in the first factor when $p = -2i$, and in the second factor when $p = -2i \pm 1$. Hence the coefficients $A_{i,\,-2i}$, $A_{i,\,-2i\pm1}$ will have their orders lowered by at least one power of m due to this cause. Further, they cannot have their orders lowered by more than one power of m from these denominators, since the coefficient of m cannot vanish while that independent of m vanishes for any values except when $i = 0$, $p = \pm 1$, a case which does not come in here. Owing to the fact, however, that any coefficient $A_{i,\,p}$ will depend on those of lower orders, this rule does not give the lowest order. It can be easily seen from the equations (4) what the order of any coefficient is. In particular, we have

$$\left.\begin{array}{ll} p = +1 & i = -1 \\ p = -1 & i = +1 \end{array}\right\} A_{i,\,-1},\ A_{-i,\,1} \text{ of order } em^{2i-1}, \qquad (i = 1, 2\ldots\infty)$$
$$\begin{array}{ll} p = +2 & i = -1 \\ p = -2 & i = +1 \end{array} \begin{array}{l} \left.\right\rbrace A_{i,\,-2},\ A_{-i,\,2} \text{ of order } e^2m^{2i-2}, \qquad (i = 2, 3\ldots\infty) \\ \left.\right\rbrace A_{1,\,-2},\ A_{-1,\,2} \text{ of order } e^2m, \end{array}$$

and so on. These are well known, but the simplicity with which they are obtained seems worth mentioning.

Returning to numerical values, the equations

$$2i + p \times 1.071583\ldots = 0,$$
$$2i + p \times 1.071583\ldots = \pm\sqrt{1 - 2m + m^2/2}$$
$$= \pm\, 1.076303\ldots$$

solved *approximately* in integers will give those coefficients whose expressions have small denominators. The set of values

$$p = -2i \text{ or } -2i \pm 1, \qquad (i = +\infty\ldots-\infty)$$

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* Comptes Rendus, t. LXXIV, p. 19.

† Acta Math., Vol. VIII., p. 36.

will as before give a series of such coefficients, but the denominators increase with the value of i. Again

$$p = \pm 15, \quad i = \mp 8$$

will be the smallest denominator of another series from the first equation and

$$p = \pm 14, \quad i = \mp 8$$

from the second equation. In this way it is possible to find those terms with comparatively large coefficients.

If we had included the terms due to parallax, solar eccentricity and inclination, the denominators would have been of the form

$$(i + cp + gq + rm)[2(i + cp + gq + rm)^2 - 2 - 4m + m^2],$$

where i, p, q, r receive all positive and negative integral values, and where g is the relation of the motion of the node to the difference of the mean motions. The values for which the first factor becomes small are well known, since it occurs directly in integrating the equations used in the older methods. The equation.

$$i + cp + gq + rm = \pm\sqrt{1 - 2m + m^2/2}$$

will have approximate solutions in integers, and those solutions in which i, p, q, r are not very large will determine coefficients great in comparison with their apparent order. These trial solutions would be obtained by inserting the numerical values of c, g, m.

The terms obtained by making the first factor small are the well-known long-period terms; those obtained from the second factor have periods of nearly a month. This closer approximation than the value unity, usually taken instead of the expression $\sqrt{1 - 2m + m^2/2}$, might reveal new terms hitherto neglected. The monthly differences between observation and the theoretical tables are still sufficiently large to justify an examination of these short-period inequalities.

III.

Determination of the Coefficients dependent on the First Power of the Eccentricity.

It has been shown that $A_{i,p}$ is at least of the order e^p in e. In order therefore to obtain the values of the coefficients depending on the first power of e only, p will have the values $+1$ and -1, while the corresponding values

which q can take are $+1$, 0 and 0, -1. Put for convenience

$$A_{i,0}=a_i,^* \quad A_{i,1}=\varepsilon_i, \quad A_{i,-1}=\varepsilon_i',$$

the equations (4) giving ε_i and ε'_{-i} will become

$$\left.\begin{aligned}&\Sigma_j\{(j,\,i,\,1,\,1)\,\varepsilon_j a_{j-i}+(j+i,\,i,\,1,\,0)\,\varepsilon_j' a_{j+i}\\ &\qquad\qquad+2\,(i,\,1)\,\varepsilon_j a_{i-j-1}+2\,[i,\,1]\,\varepsilon_j' a_{-i-j-1}\}=0,\\ &\Sigma_j\{(j-i,\,i,\,-1,\,0)\,\varepsilon_j a_{j-i}+(j,\,-i,\,-1,\,-1)\,\varepsilon_j' a_{j+i}\\ &\qquad\qquad+2\,[-i,\,-1]\,\varepsilon_j a_{i-j-1}+2\,(-i,\,-1)\,\varepsilon_j' a_{-i-j-1}\}=0.\end{aligned}\right\}\quad(5)$$

Since the quantities a_i are known, these equations will suffice to determine the relations of ε_i and ε'_{-i} to ε_0 or ε_0'. There are in all as many equations as there are unknowns ε_i, ε'_{-i}. In order therefore that these equations may be consistent, some relation must hold between the quantities a_i and the coefficients; it is from this relation that the value of c is to be found. For the sake of the argument which follows, the equations will be given another form.

In (3) put $p=1$, $q=0$, 1, add and subtract the results. We obtain

$$\left.\begin{aligned}&\Sigma_j\{c^2 a_{j-i}+G_{j,i}\,a_{j-i}+G_{j+i,i}\,a_{j+i}+\tfrac{3}{4}m^2(a_{i-j-1}+5a_{-i-j-1})\}\,X_j\\ &\qquad\qquad\qquad+c\Sigma_j\,(H_{j,i}\,a_{j-i}-H_j' a_{j+i})\,Y_j=0,\\ &\Sigma_j\{c^2 a_{j-i}+G_{j,i}\,a_{j-i}-G_{j+i,i}\,a_{j+i}+\tfrac{3}{4}m^2(a_{i-j-1}-5a_{-i-j-1})\}\,Y_j\\ &\qquad\qquad\qquad+c\Sigma_j\,(H_{j,i}\,a_{j-i}+H_j' a_{j+i})\,X_j=0,\end{aligned}\right\}\quad(6)$$

where

$$\begin{aligned}&G_{j,i}=\tfrac{1}{2}(2j+1)^2+(2j+1)(i+2m)+\tfrac{9}{4}m^2,\\ &H_{j,i}=\tfrac{3}{2}(2j+1)+i+2m, \qquad H_j'=\tfrac{1}{2}(2j+1),\\ &X_j\;=\varepsilon_j+\varepsilon_j', \qquad\qquad\qquad\quad Y_j=\varepsilon_j-\varepsilon_j'.\end{aligned}$$

Let ∞ denote the number of integers in the sequence $1,\,2,\,3\,\ldots\,\infty$, there are then $4\times\infty+2$ of these equations and the same number of unknowns X_j and Y_j. As these equations are linear, suppose the unknowns eliminated by a determinant, which may be looked upon as in equation in c having $4\times\infty+2$ roots. It is immediately evident from its form that it involves only even powers of c, since the coefficients G, H, H' do not contain c.

The above equations are convergent series when $|c^2 Y_j|$, $|c^2 X_j|$ are not infinite for any value of j, and the series $\Sigma|a_i|$ is absolutely convergent.

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* The coefficient here called a_i is therefore the same as Dr. Hill's a_i/a_0. The change is convenient, since we are principally concerned with the ratios of the coefficients to a_0; a_i will be a numerical quantity ultimately.

In order to find what values of c will make these equations consistent, it is possible to proceed as follows. In either of equations (3) put $c + 2s$ for c and $i - s$ for i, where s is an integer, the equation considered will remain unaltered. Hence the result of the first substitution causes the equation with suffix i to become that with suffix $i + s$. Any such change therefore merely permutes the equations into one another. Hence if c_0 be any value of c which renders the equations consistent, any one of the set of values

$$\pm c_0, \quad \pm (c_0 \pm 2), \quad \pm (c_0 \pm 4), \quad \pm (c_0 \pm 6), \ldots$$

will have the same effect. If we form a determinant by eliminating the X_j and Y_j, and if this determinantal equation for c have one root c_0, it will have as roots the set of values given above. This determines $2 \times \infty + 1$ values of c^2 when c_0 is known. As there are $4 \times \infty + 2$ values altogether, there are two values of c^2 which are both non-infinite and which do not differ by any even integer. It may be shown that there is one value of c_0^2 which is zero. Dr. Hill,* in obtaining the other value of c_0^2, submitted equations (1) to a variation δ and obtained two equations linear in δu, δs. A solution of these equations might have been obtained by putting

$$\delta u = \Sigma_i \varepsilon_i \zeta^{2i+1+c} + \Sigma_i e_i' \zeta^{2i+1-c}$$

and the corresponding value for δs, that is the solution considered above when the second power of the eccentricity is neglected. He points out that

$$\delta u = Du, \quad \delta s = Ds,$$

where u, s represent the second particular solution is also a solution of the equations. Now Du, Ds are odd power series in ζ, and therefore the above solution demands that c shall have an even integral value. We thus obtain the result that the roots of the determinant in c^2 are

$$\left.\begin{array}{l} c_0^2, \ (c_0 \pm 2)^2, \ (c_0 \pm 4)^2, \ldots . (c_0 \pm 2i)^2, \ldots . \\ 0^2, \ 2^2, \ 2^2, \ 4^2, \ 4^2, \ldots . (2i)^2, \ (2i)^2, \ldots . \end{array}\right\} i = 1, 2 \ldots . \infty .$$

The second series of values does not appear in Dr. Hill's determinant. He puts

$$\delta u = v . Du, \quad \delta s = w . Ds$$

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* On the Part of the Motion of the Lunar Perigee, etc., Acta Math., Vol. VIII.

and forms the equations in v, w; these are reduced so that v and w can both be expressed by a single equation

$$D^2\mathrm{w} = \Theta\mathrm{w},$$

where Θ is a known function. The determinant which is ultimately formed then consists of an equation in c^2 which has only $2 \times \infty + 1$ roots, comprising those of the first series above.

It might seem that it was now possible to proceed to obtain the value of c_0 as Dr. Hill has done in the memoir just referred to. When this is attempted, however, we are met at the outset by the difficulty that the determinant is not convergent. The coefficient of the highest power of c^2 in it is the square of

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots\; a_{-1} & 1 & a_1 & a_2 & a_3 \;\cdots \\ \cdots\; a_{-2} & a_{-1} & 1 & a_1 & a_2 \;\cdots \\ \cdots\; a_{-3} & a_{-2} & a_{-1} & 1 & a_1 \;\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \equiv \nabla,$$

which does not satisfy the second condition given by M. Poincaré, namely, that the sum of the absolute values of the non-diagonal elements should be finite.* Also it does not seem possible to find any set of factors which would render this determinant at once finite and adapted to calculation.

Let $\Delta(c)$ denote the determinant form obtained from equations (6). We have

$$(\cos \pi c - 1)(\cos \pi c - \cos \pi c_0) \equiv \Delta'(c) \div \nabla^2,$$

where $\Delta'(c)$ has been deduced from $\Delta(c)$ by suitably multiplying the rows of this latter so as to make the coefficients of the highest power of c^2 on each side of the identity the same. The determinants $\Delta'(c)$ and ∇ are not finite, and it does not seem possible to find any set of factors to make them both so. All that can be said at present is this: as c_0 is known to have a determinate and finite value (in the case of our moon), the ratio $\Delta'(c)/\nabla^2$ is not infinite for finite values of c and will not be zero unless the value given to c be one of the roots of the equation

$$(\cos \pi c - 1)(\cos \pi c - \cos \pi c_0) = 0.$$

For the calculation of c_0 this method appears to be of no use, but the algebraical forms which it brings forward are not devoid of interest.

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* Bulletin de la Société mathématique de France, t. XIV, pp. 77–90.

The value of c can be calculated by continued approximation in connection with the coefficients ε_i, ε_i' when we know a first approximation to it. It is easily seen that the value we want is nearly unity; from equations (5) this would give as a real first approximation

$$c = +\sqrt{\left(1 - 2m + \frac{m^2}{2}\right)}.$$

As a calculation by this method would involve recomputing the coefficients in (5) at each step, the method would be exceedingly laborious. I have therefore thought it best, instead of attempting to find c, to assume Dr. Hill's value and calculate the corresponding values of ε_i and ε_i' in terms of ε_0 and ε_0'. There are then unused the two equations for ε_0 and ε_0', which by substitution of the values of ε_i and ε_i' should give the same ratio $\varepsilon_0/\varepsilon_0'$ if the assumed value of c is correct. Since the equations are all linear, this is not difficult, and such a proceeding will give a real verification of the value of c as far as the approximations are taken.

IV.

Calculation of the Coefficients ε_i and ε_i'.

In order to calculate the coefficients in the equations (5), it is convenient to put them into that form which will render the labor as small as possible. Let

$$\left.\begin{matrix} L_i \\ L_i' \end{matrix}\right\} = -\frac{(2i \pm c)^2 - (2 + 4m - m^2) + (2i \pm c)(2 + 2m)}{(2i \pm c)\{2(2i \pm c)^2 - 2 - 4m + m^2\}},$$

$$\left.\begin{matrix} M_i \\ M_i' \end{matrix}\right\} = -\frac{2i \pm c - 2 - 2m}{(2i \pm c)\{2(2i \pm c)^2 - 2 - 4m + m^2\}},$$

where L_i, M_i refer to the upper signs and L_i', M_i' to the lower. We then have

$$f(i, j, \ \ 1, q) = (2j + qc)\, L_i + (2j + qc)^2 M_i,$$
$$f(i, j, -1, q) = (2j + qc)\, L_i' + (2j + qc)^2 M_i'.$$

When we neglect all terms except those depending on the first power of the eccentricity, q takes the values 1, 0 in the first of these expressions and the values 0, -1 in the second; and the first terms in equations (5) take the forms

$$\Sigma_j \left[P_{j,\,i} a_{j-i} \varepsilon_j + Q_{j,\,i} a_j \varepsilon_{j-i}'\right],$$
$$\Sigma_j \left[Q_{j,\,i}' a_{j-i} \varepsilon_j + P_{j,\,i}' a_j \varepsilon_{j-i}'\right],$$

where

$$P_{j,\,i} = (2j + c)\,L_i + (2j + c)^2 M_i, \qquad Q_{j,\,i} = 2jL_i + 4j^2 M_i,$$
$$P'_{j,\,i} = (2j - c)\,L'_i + (2j - c)^2 M_i, \qquad Q'_{j,\,i} = 2jL'_i + 4j^2 M_i.$$

These forms are given *in extenso* because the same transformations have been used to obtain the whole set of coefficients $A_{p,\,i}$ calculated. The quantities L_i, M_i, L'_i, M'_i are calculated once for all for each equation; it is then easy to obtain each coefficient $P_{j,\,i}$, etc. Their values are verified by the known results $P_{i,\,i} = P'_{i,\,i} = -1$. The coefficients once calculated, the work proceeds in the usual way by successive approximation, ε_i and ε'_i being found as linear functions of the unknowns ε_0 and ε'_0.

The coefficients a_i are, as has been mentioned, the same as Dr. Hill's a_i/a_0, and the numerical values of them are taken from his paper (Vol. I, p. 247).

It is useful to have an equation of verification at each step. Either of equations (6) or the second of equations (2) answer this purpose. One of equations (6) with suffix i enables us to verify ε_i, ε'_{-i}, ε_{-i} and ε'_i at once; and since the result of substituting the values obtained ought to render it an identity, ε_0 can be put equal to ε'_0 for this purpose. In this way the following table of results has been verified:

	ε_{-1}		ε'_1
1st approx.	$+.01998\ 37455\,\varepsilon_0 + .20559\ 07151\,\varepsilon'_0$		$-.01054\ 25540\,\varepsilon_0 - .07777\ 75955\,\varepsilon'_0$
2d "	$+.00001\ 51270\,\varepsilon_0 + .00008\ 82876\,\varepsilon'_0$		$-.00000\ 42515\,\varepsilon_0 - .00001\ 79511\,\varepsilon'_0$
3d "	$+.00000\ 00038\,\varepsilon_0 + .00000\ 00085\,\varepsilon'_0$		$-.00000\ 00003\,\varepsilon_0 + .00000\ 00036\,\varepsilon'_0$
$\varepsilon_{-1} =$	$+.01999\ 88763\,\varepsilon_0 + .20567\ 90112\,\varepsilon'_0$	$\varepsilon'_1 =$	$-.01054\ 68058\,\varepsilon_0 - .07779\ 55430\,\varepsilon'_0$

	ε_1		ε'_{-1}
1st approx.	$+.00308\ 02608\,\varepsilon_0 - .00092\ 87880\,\varepsilon'_0$		$-.00108\ 65604\,\varepsilon_0 - .00019\ 44186\,\varepsilon'_0$
2d "	$+.00000\ 00319\,\varepsilon_0 + .00000\ 07811\,\varepsilon'_0$		$-.00000\ 00355\,\varepsilon_0 - .00000\ 15807\,\varepsilon'_0$
3d "	$+.00000\ 00000\,\varepsilon_0 + .00000\ 00002\,\varepsilon'_0$		$-.00000\ 00001\,\varepsilon_0 - .00000\ 00006\,\varepsilon'_0$
$\varepsilon_1 =$	$+.00308\ 02927\,\varepsilon_0 - .00092\ 80067\,\varepsilon'_0$	$\varepsilon'_{-1} =$	$-.00108\ 65960\,\varepsilon_0 - .00019\ 59999\,\varepsilon'_0$

	ε_{-2}		ε'_2
1st approx.	$+.00001\ 15195\,\varepsilon_0 + .00007\ 34781\,\varepsilon'_0$		$-.00005\ 93615\,\varepsilon_0 - .00043\ 19502\,\varepsilon'_0$
2d "	$+.00000\ 00010\,\varepsilon_0 - .00000\ 00090\,\varepsilon'_0$		$-.00000\ 00261\,\varepsilon_0 - .00000\ 01280\,\varepsilon'_0$
$\varepsilon_{-2} =$	$+.00001\ 15205\,\varepsilon_0 + .00007\ 34691\,\varepsilon'_0$	$\varepsilon'_2 =$	$-.00005\ 93876\,\varepsilon_0 - .00043\ 20782\,\varepsilon'_0$

	ε_2	ε'_{-2}
1st approx.	$+.00001\ 47373\,\varepsilon_0 - .00000\ 85454\,\varepsilon'_0$	$+.00000\ 01043\,\varepsilon_0 - .00000\ 08608\,\varepsilon'_0$
2d "	$+.00000\ 00003\,\varepsilon_0 + .00000\ 00076\,\varepsilon'_0$	$+.00000\ 00000\,\varepsilon_0 - .00000\ 00010\,\varepsilon'_0$

$$\varepsilon_2 = +.00001\ 47376\,\varepsilon_0 - .00000\ 85378\,\varepsilon'_0 \quad \varepsilon'_{-2} = +.00000\ 01043\,\varepsilon_0 - .00000\ 08618\,\varepsilon'_0$$

$$\varepsilon_{-3} = -.00000\ 00193\,\varepsilon_0 - .00000\ 01734\,\varepsilon'_0 \quad \varepsilon'_3 = -.00000\ 04039\,\varepsilon_0 - .00000\ 29218\,\varepsilon'_0$$

$$\varepsilon_3 = +.00000\ 00843\,\varepsilon_0 - .00000\ 00708\,\varepsilon'_0 \quad \varepsilon'_{-3} = +.00000\ 00024\,\varepsilon_0 - .00000\ 00055\,\varepsilon'_0$$

$$\varepsilon_{-4} = -.00000\ 00001\,\varepsilon_0 - .00000\ 00012\,\varepsilon'_0 \quad \varepsilon'_4 = -.00000\ 00029\,\varepsilon_0 - .00000\ 00212\,\varepsilon'_0$$

A second approximation produces no effect in the last three coefficients. The advantage in point of accuracy is easily seen by the speed with which the coefficients approximate.

In equations (6) put $i = 0$; they become

$$\Sigma_j\left[\left\{c^2 + (2j+1+2m)^2 + \tfrac{1}{2}m^2\right\}a_j + \tfrac{3}{2}m^2a_{-j-1}\right]X_j + c\Sigma_j(2j+1+2m)\,a_jY_j = 0,$$
$$\Sigma_j\left[c^2a_j - 3m^2a_{-j-1}\right]Y_j \qquad + c\Sigma_j(2j+2+2m)\,a_jX_j = 0.$$

Remembering that

$$X_j = \varepsilon_j + \varepsilon'_j, \quad Y_j = \varepsilon_j - \varepsilon'_j,$$

these give on substitution of the values just obtained for ε_j, ε'_j,

$$2.01291\ 56632\ 7\ X_0 + 1.24333\ 41788\ 2\ Y_0 = 0,$$
$$2.31465\ 02085\ 6\ X_0 + 1.14989\ 92484\ 3\ Y_0 = 0,$$

whence

$$\frac{Y_0}{X_0} = -\ 2.01291\ 56632\ 7,$$
$$\frac{Y_0}{X_0} = -\ 2.01291\ 56634\ 5.$$

The difference is probably due to accumulated errors. Taking the mean of these two values,

$$\frac{Y_0}{X_0} = -\ 2.01291\ 56633\ 6.$$

Hence with an error not exceeding one unit in the last place given,

$$\frac{X_0}{Y_0} = -\ .49679\ 18022.$$

We may therefore say, from the way that c is involved in the equations given above, *that the assumed value is certainly correct to the tenth place of decimals.*

As only one of the symbols ε_0, ε'_0, X_0, Y_0 can be arbitrary, we must decide which is the most convenient. Below it is shown that if e be the value of the eccentricity which Delaunay has used, Y_0 differs very little indeed from $2e$. In the case of our moon, Y_0 will therefore be just over .1, a convenient number to deal with. Hence Y_0 seems the best suited to be the arbitrary constant, and all the coefficients will be expressed in terms of it.

From the ratio X_0/Y_0 we find the ratios ε_0/Y_0, ε'_0/Y_0, and substituting in the table given above, we obtain as the final values of the coefficients the following:

$$\frac{\varepsilon_0}{Y_0} = +.25160\ 40989, \qquad \frac{\varepsilon'_0}{Y_0} = -.74839\ 59011,$$

$$\frac{\varepsilon_{-1}}{Y_0} = -.14889\ 75297, \qquad \frac{\varepsilon'_1}{Y_0} = +.05556\ 82459,$$

$$\frac{\varepsilon_{+1}}{Y_0} = +.00146\ 95307, \qquad \frac{\varepsilon'_{-1}}{Y_0} = -.00012\ 67065,$$

$$\frac{\varepsilon_{-2}}{Y_0} = -.00005\ 20854, \qquad \frac{\varepsilon'_2}{Y_0} = +.00030\ 84234,$$

$$\frac{\varepsilon_2}{Y_0} = +.00001\ 00977, \qquad \frac{\varepsilon'_{-2}}{Y_0} = +.00000\ 06713,$$

$$\frac{\varepsilon_{-3}}{Y_0} = +.00000\ 01250, \qquad \frac{\varepsilon'_3}{Y_0} = +.00000\ 20851,$$

$$\frac{\varepsilon_3}{Y_0} = +.00000\ 00742, \qquad \frac{\varepsilon'_{-3}}{Y_0} = +.00000\ 00048,$$

$$\frac{\varepsilon_{-4}}{Y_0} = +.00000\ 00009, \qquad \frac{\varepsilon'_4}{Y_0} = +.00000\ 00243.$$

Let $$D = (n - n')(t - t_1), \qquad l = c\,(n - n')(t - t_0),$$

and let v denote the excess of the true over the mean longitude.

The values just obtained give for the portion depending only on Y_0,

$$\begin{aligned} r\cos v = a_0 Y_0 [& -.49679\ 18022 \cos l \\ & -.09332\ 92838 \cos(2D - l) + .00134\ 28242 \cos(2D + l) \\ & +.00025\ 63380 \cos(4D - l) + .00001\ 07690 \cos(4D + l) \\ & +.00000\ 22101 \cos(6D - l) + .00000\ 00790 \cos(6D + l) \\ & +.00000\ 00252 \cos(8D - l) + .00000\ 00005 \cos(8D + l) \\ & +.00000\ 00003 \cos(10D - l)], \end{aligned}$$

$$
\begin{aligned}
r\cos v = a_0 Y_0 [&+ 1.00000\ 00000 \sin l \\
&+ .20446\ 57756 \sin(\ 2D - l) + .00159\ 62372 \sin(2D + l) \\
&+ .00036\ 05088 \sin(\ 4D - l) + .00000\ 94264 \sin(4D + l) \\
&+ .00000\ 19601 \sin(\ 6D - l) + .00000\ 00694 \sin(6D + l) \\
&+ .00000\ 00234 \sin(\ 8D - l) + .00000\ 00004 \sin(8D + l) \\
&+ .00000\ 00003 \sin(10D - l)].
\end{aligned}
$$

The coefficients of the terms with arguments $8D + l$ and $10D - l$ have been added by induction.

Dr. Hill suggested (Vol. I, p. 145) that a_0 should play the part usually assigned to a in the Lunar Theory. In the same way it seems best that Y_0 should be regarded as the arbitrary constant of this solution. The above then is complete so far as the first power of Y_0 is concerned.

It will be useful to compare these expressions with the corresponding ones obtained by Delaunay.* For this purpose the values found above must be transferred to polar coordinates. We have

$$\tan v = y'/x'.$$

In y' and x' are included the terms depending on m only. Denote by the operator δ the aggregate of the new terms added, depending on the first power of Y_0, then

$$\delta v = \frac{x'\delta y' - y'\delta x'}{x'^2 + y'^2}.$$

In this formula x', y' denote the values of $r\cos v$, $r\sin v$ when Y_0 is neglected, and $\delta x'$, $\delta y'$ the values when the terms depending only on Y_0 are taken into account. The value of δv is most easily found by giving to $2D$ the values 0, 30, 90, 150, 180 successively and calculating the coefficients of $\sin l$ and $\cos l$ in the several cases. In this way are obtained the series of coefficients in longitude. That of $\sin l$ comes out to be

$$.99972\ 87063\ Y_0.$$

Delaunay† fixes the coefficient of $\sin l$ as that which would be obtained in a purely elliptic motion. To the first power of the eccentricity this is $2e$. Hence

$$2e = .99972\ 87063\ Y_0,$$

and therefore

$$Y_0 = 2.00054\ 2735\quad e.$$

* Mémoires de l'Académie des Sciences, t. XXIX, pp. 818, 821, etc.

† Delaunay, Mém. cit., p. 798.

Using Delaunay's value of e, namely,

$$e = .05489\ 930,$$

we obtain the following series of coefficients in longitude:

$$+ 4607''.984 \sin(2D - l) + 35''.2200 \sin(4D - l)$$
$$+ 0''.2906 \sin(6D - l) + 0''.0027 \sin(8D - l)$$
$$+ 174''.8610 \sin(2D + l) + 1''.4460 \sin(4D + l)$$
$$+ 0''.0121 \sin(6D + l) + 0''.0001 \sin(8D + l).$$

With the assumed value of e, these are correct to the last place of decimals given. The coefficient of $\sin(8D + l)$ is added by induction. Delaunay's values for the corresponding parts are

$$+ 4607.771 \sin(2D - l) + 35''.1542 \sin(4D - l) + 0''.2174 \sin(6D - l)$$
$$+ 174''.8660 \sin(2D + l) + 1''.4094 \sin(4D + l) + 0''.0055 \sin(6D + l).$$

[*To be continued.*]

HAVERFORD COLLEGE, *December*, 1892.

On the Transformation of Linear Differential Equations of the Second Order with Linear Coefficients.

BY OSKAR BOLZA.

The transformation and reduction to a canonical form of the linear differential equation of the second order,

$$(A_0x + B_0)\frac{d^2y}{dx^2} + (A_1x + B_1)\frac{dy}{dx} + (A_2x + B_2)y = 0 \tag{A}$$

has been studied by Weiler* and Schlömilch,† and again recently by Pochhammer,‡ in connection with his investigations on the integration of linear differential equations by means of definite integrals.

In the following pages I take up the problem anew, treating it, however, by methods of the *Theory of Invariants.*

The first thing necessary is, then, to find a *group of transformations* which transforms the given differential equation into one of the same type. For this purpose it is preferable to substitute for the differential equation (A) the two differential equations,§

$$\frac{d^2y}{dx^2} + 2\left(\frac{a}{x} + b\right)\frac{dy}{dx} + \left(\frac{f}{x^2} + \frac{2g}{x} + h\right)y = 0, \tag{B}$$

$$\frac{d^2y}{dx^2} + 2(a + bx)\frac{dy}{dx} + (f + 2gx + hx^2)y = 0, \tag{C}$$

which, combined, comprise the differential equation (A) as a special case.

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* Crelle's Journal, Vol. 51 (1856), p. 123.

† Compendium der höheren Analysis: Braunschweig, 1879, Vol. II, p. 523.

‡ Mathematische Annalen, Vol. 38 (1891), pp. 225, 241.

§ This is the form in which Weiler considers the problem.

For if $A_0 \neq 0$, (A) can be reduced to the type (B) by the substitution $A_0 x + B_0 = x'$; and if $A_0 = 0$, (A) is of the type (C).

The integrals of (A) are of a very different analytical character according as A_0 is $\neq 0$ or $= 0$; it seems, therefore, indeed more natural to separate the essentially different cases rather than to artificially unite them in one canonical form.

§1.

The Differential Equation (B).

a). Group of Transformations.

The differential equation (B) is transformed by the substitution

$$x = \varkappa x', \quad y = \nu x'^{\lambda} e^{\mu x'} y', \quad (\varkappa \neq 0) \tag{1}$$

into a differential equation of the same type, viz.

$$\frac{d^2 y'}{dx'^2} + 2\left(\frac{a'}{x'} + b'\right)\frac{dy'}{dx'} + \left(\frac{f'}{x'^2} + \frac{2g'}{x'} + h'\right) y' = 0, \tag{B'}$$

where

$$\left.\begin{aligned} a' &= a + \lambda, \\ b' &= \varkappa b + \mu, \\ f' &= \lambda(\lambda - 1) + 2a\lambda + f, \\ g' &= \varkappa g + \varkappa\lambda b + \mu a + \lambda\mu, \\ h' &= \varkappa^2 h + 2\varkappa\mu b + \mu^2. \end{aligned}\right\} \tag{2}$$

The entire system of all transformations of the form (1) constitute a *group* $\mathfrak{G}$; to every transformation of the group there exists an "inverse" transformation in the group. Two differential equations (B) and (B′) will be said to be *equivalent* with respect to the group $\mathfrak{G}$, if they can be transformed into one another by transformations of the group. In order that (B) and (B′) shall be equivalent, it is necessary and sufficient that there exist values of the parameters $\varkappa$, λ, μ which satisfy the system (2), and that moreover $\varkappa \neq 0$.

b). Invariants.

The elimination of λ, μ between the equations (2) leads to the following result:

The Differential Equation (B) *has, with respect to the group* $\mathfrak{G}$, *an* ABSOLUTE INVARIANT,

$$A = f + a - a^2 \tag{3}$$

and TWO INVARIANTS,

$$\left.\begin{aligned} B &= g - ab, \\ C &= h - b^2. \end{aligned}\right\} \tag{4}$$

Their invariant character is expressed by the equations

$$\left.\begin{aligned} A' &= A, \\ B' &= \varkappa B, \\ C' &= \varkappa^2 C, \end{aligned}\right\} \tag{5}$$

if A', B', C' denote the same expressions formed with the coefficients of the transformed differential equation (B′).

c). *Canonical forms and conditions of equivalence.*

The transformation

$$x = \varkappa z, \quad y = z^{-a} e^{-\varkappa b z} v \tag{6}$$

of our group reduces (B) to

$$\frac{d^2 v}{dz^2} + \left(\frac{A}{z^2} + \frac{2\varkappa B}{z} + \varkappa^2 C\right) v = 0. \tag{7}$$

Hence we obtain, by a proper choice of $\varkappa$, the following canonical forms:

I. *Case:* $B \neq 0$, $C \neq 0$.

$$\frac{d^2 v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z} + I\right) v = 0, \tag{8}$$

where I denotes the absolute invariant

$$I = \frac{C}{B^2}. \tag{9}$$

II. *Case:* $B \neq 0$, $C = 0$.

$$\frac{d^2 v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z}\right) v = 0. \tag{10}$$

III. *Case:* $B = 0$, $C \neq 0$.

$$\frac{d^2 v}{dz^2} + \left(\frac{A}{z^2} + 1\right) v = 0. \tag{11}$$

IV. *Case:* $B=0,\ C=0$.

$$\frac{d^2v}{dz^2}+\frac{A}{z^2}=0. \tag{12}$$

Hence follows the theorem:

In order that two differential equations (B) *and* (B′) *shall be equivalent, it is necessary and sufficient that*

$a)\quad A'=A$

and that $b)$ *a quantity* $\varkappa \neq 0$ *can be so determined that*

$$B'=\varkappa B,$$
$$C'=\varkappa^2 C.$$

d). *Other canonical forms, preferable for integration by series.*

From the general theory of linear differential equations it follows that (B) can be integrated by a permanently convergent *power-series* of the form

$$y=\sum_{\nu=0}^{\infty} c_\nu x^{r+\nu}, \qquad (c_0 \neq 0),$$

where r is a root of the *indicial equation*,*

$$f_0(r)=r(r-1)+2ra+f=0. \tag{13}$$

In passing, we notice that its discriminant is an absolute invariant, viz.

$$\Delta=\frac{1}{4}-A. \tag{14}$$

The coefficients c_ν are determined by the recurrent formula†

$$c_\nu f_0(r+\nu)+c_{\nu-1}f_1(r+\nu-1)+c_{\nu-2}f_2(r+\nu-2)=0, \tag{15}$$

where

$$f_1(r)=2(g+br), \quad f_2(r)=h.$$

We propose to use the transformation (1) to reduce the relation (15) to only *two terms*. This can always be obtained by making $h'=0$ by a proper choice

* "Determinierende Fundamentalgleichung (Fuchs); see Craig, A Treatise on Linear Differential Equations, p. 118.

† Fuchs, Crelle's Journal, Vol. 66; Frobenius, Crelle's Journal, Vol. 76.

of μ; at the same time we can choose λ so that $f'=0$. This leads to the following canonical forms:

a). *Case* I *and* III: $C \neq 0$.

$$\xi \frac{d^2\eta}{d\xi^2} = (\xi - \rho)\frac{d\eta}{d\xi} + \alpha\eta, * \tag{16}$$

where

$$\rho = 1 + \sqrt{1-4A}, \quad \alpha = \frac{1}{2} + \frac{1}{2}\sqrt{1-4A} - \frac{B}{\sqrt{-C}}.$$

b). *Case* II: $B \neq 0, \ C = 0$.

$$\xi \frac{d^2\eta}{d\xi^2} + \rho\frac{d\eta}{d\xi} - \eta = 0. \dagger \tag{17}$$

c). *Case* IV: $B = 0, \ C = 0$.

$$\xi \frac{d^2\eta}{d\xi^2} + \rho\frac{d\eta}{d\xi} = 0. \tag{18}$$

In (17) and (18) ρ has the same value as in (16).

The reduction of (15) to two terms may, however, be performed still in other ways which offer certain advantages.

If $C \neq 0$, we can determine λ and μ so that

$$g' = 0 \text{ and } h' = 0,$$

since the resultant obtained by eliminating μ is

$$C(\lambda + a)^2 + B^2 = 0.$$

Hence the canonical form

$$\frac{d^2u}{dt^2} + \left(\frac{1-\lambda'-\lambda''}{t} - 1\right)\frac{du}{dt} + \frac{\lambda'\lambda''}{t^2}u = 0, \tag{19}$$

$$\left.\begin{matrix}\lambda' \\ \lambda''\end{matrix}\right\} = \frac{1}{2} \div \frac{B}{\sqrt{-C}} \pm \frac{1}{2}\sqrt{1-4A}.$$

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* This is, apart from the notation, Weiler's canonical form; see l. c., p. 127; also Schlömilch, l. c., p. 531; and Pochhammer, Math. Ann., Vol. 36, p. 84.

† Pochhammer, Math. Ann., Vol. 38, p. 226.

This canonical form is symmetric with respect to the two fundamental integrals; they are, if $\lambda' - \lambda''$ is not an integer,

$$\left.\begin{aligned} u_1 &= t^{\lambda'} F(\lambda'\,;\ 1+\lambda'-\lambda''\,;\ t),\\ u_2 &= t^{\lambda''} F(\lambda''\,;\ 1+\lambda''-\lambda'\,;\ t),\end{aligned}\right\} \tag{20}$$

where we denote, with Pochhammer,* by $F(a\,;r\,;t)$ the permanently convergent series

$$F(a\,;\ r\,;\ t) = 1 + \frac{a}{1.r}\,t + \frac{a(a+1)}{1.2\,r(r+1)}\,t^2 + \dots. \tag{21}$$

The two canonical forms (16) and (19) are transformable into one another by the substitution

$$t = \xi, \quad u = t^{\lambda'}\eta.$$

If, in particular, $B = 0$, while as before $C \neq 0$, the relation (13) may be reduced to two terms by making

$$f_1(r) \equiv 0.$$

For, after having determined μ so that $b' = 0$, it follows from $B = 0$ that also $g' = 0$. Moreover, the parameter λ may be used to make $a' = \frac{1}{2}$ or to make $f' = 0$. Accordingly we obtain the two equivalent canonical forms

$$\frac{d^2u}{dt^2} + \frac{1}{t}\,\frac{du}{dt} + \left(1 - \frac{n^2}{t^2}\right)u = 0, \tag{22}$$

(*Bessel's Equation.*)

or

$$\frac{d^2u}{dt^2} + \frac{2n+1}{t}\,\frac{du}{dt} + u = 0, \tag{23}$$

$$n^2 = \frac{1}{4} - A.$$

In concluding we give the necessary and sufficient condition that our differential equation (B) shall be reducible, by a transformation of our group, to the form

$$\frac{d^2y'}{dx'^2} + 2\left(\frac{a'}{x'} + b'\right)\frac{dy'}{dx'} = 0, \tag{24}$$

in which case it is integrable by quadratures; the condition is

$$(AC - B^2)^2 + B^2C = 0. \tag{25}$$

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* Math. Annalen, Vol. 36, p. 84.

§2.

The Differential Equation (*C*).

a). *Group of transformations.*

The differential equation

$$\frac{d^2y}{dx^2} + 2(a+bx)\frac{dy}{dx} + (f + 2gx + hx^2)\,y = 0 \tag{C}$$

is transformed by the transformation

$$x = \varkappa x' + \lambda, \quad y = e^{\mu x' + \frac{\nu}{2}x'^2}y', \qquad (\varkappa \neq 0) \tag{1}$$

into

$$\frac{d^2y'}{dx'^2} + 2(a'+b'x')\frac{dy'}{dx'} + (f' + 2g'x' + h'x'^2)\,y' = 0, \tag{C'}$$

where

$$\left.\begin{aligned}
a' &= \varkappa\,(a+b\lambda) + \mu,\\
b' &= \varkappa^2 b + \nu,\\
f' &= \varkappa^2(f + 2g\lambda + h\lambda^2) + 2\mu\varkappa\,(a+b\lambda) + \mu^2 + \nu,\\
g' &= \varkappa^3(g + \lambda h) + \mu\varkappa^2 b + \nu\varkappa\,(a+b\lambda) + \mu\nu,\\
h' &= \varkappa^4 h + 2\nu\varkappa^2 b + \nu^2.
\end{aligned}\right\} \tag{2}$$

The transformations (1) form again a group, $\mathfrak{H}$, and the same conclusions can be applied as in §1.

b). *Invariants and covariants.*

The differential equation (C) has, with respect to the group $\mathfrak{H}$, *two invariants,*

$$\left.\begin{aligned}
H &= h - b^2,\\
D &= (h-b^2)(f-a^2-b) - (g-ab)^2,
\end{aligned}\right\} \tag{3}$$

characterized by

$$H' = \varkappa^4 H, \quad D' = \varkappa^6 D. \tag{4}$$

But these covariants are not sufficient to distinguish between the non-equivalent cases. It seems necessary to resort to a *covariant-criterion.* If we denote

$$\left.\begin{aligned}
F &= f - a^2 - b, & F' &= f' - a'^2 - b',\\
G &= g - ab\quad, & G' &= g' - a'b'\quad,\\
H &= h - b^2\quad, & H' &= h' - b'^2\quad,
\end{aligned}\right\} \tag{5}$$

the function

$$\Phi(x) = F + 2Gx + Hx^2 \tag{6}$$

has the characteristic property of a covariant, viz.

$$F' + 2G'x' + H'x'^2 = \varkappa^2 (F + 2Gx + Hx^2),$$

or shorter,

$$\Phi'(x') = \varkappa^2 \Phi(x), \tag{7}$$

as is seen from the following relations:

$$\left.\begin{aligned} F' &= \varkappa^2 F + 2\varkappa^2\lambda G + \varkappa^2\lambda^2 H, \\ G' &= \varkappa^3 G + \varkappa^3\lambda H, \\ H' &= \varkappa^4 H. \end{aligned}\right\} \tag{8}$$

c). *Reduction to canonical forms and conditions of equivalence.*

By the transformation

$$y = e^{-\left(ax + b\frac{x^2}{2}\right)} v \tag{9}$$

(which is one of our group $\mathfrak{H}$), the differential equation (C) is reduced to

$$\frac{d^2v}{dx^2} + (F + 2Gx + Hx^2) v = 0. \tag{10}$$

This differential equation can be further reduced by a transformation $x = \varkappa z + \lambda$. The following canonical forms are obtained:

I. *Case:* $H \neq 0$.

$$\frac{d^2v}{dz^2} + (\rho + z^2) v = 0, \tag{11}$$

$$\rho = \sqrt{I},$$

if we denote by I the absolute invariant

$$I = \frac{D^2}{H^3}. \tag{12}$$

II. *Case:* $H=0$, $D \neq 0$.

It follows: $G \neq 0$, hence we can make $F'=0$, $G'=-\frac{1}{3}$,

$$\frac{d^2v}{dz^2}-\frac{1}{3}zv=0, \tag{13}$$

(*Scherk**–*Lobatto's*† *equation*.)

III. *Case:* $H=0$, $D=0$.

It follows: $G=0$.

1. *Subcase:* $\Phi(x) \not\equiv 0$.

It follows: $F \neq 0$, hence we may make $F'=1$,

$$\frac{d^2v}{dz^2}+v=0. \tag{14}$$

2. *Subcase:* $\Phi(x) \equiv 0$; that is, $F=0$, $G=0$, $H=0$, hence

$$\frac{d^2v}{dz^2}=0. \tag{15}$$

Hence the result:

In order that two differential equations (C) *and* (C′) *shall be equivalent, it is necessary and sufficient that*

a). *A quantity* $\varkappa \neq 0$ *can so be determined that* $H'=\varkappa^4 H$, $D'=\varkappa^6 D$.

b). *That the two covariants* $\Phi(x)$ *and* $\Phi'(x')$ *are either both identically zero, or both not identically zero.*

In case I, a different canonical form is preferable for purposes of integration by series or by definite integrals. If $H \neq 0$, $\varkappa$, λ, μ, ν can be so determined that

$$a'=0,\ g'=0,\ h'=0,\ b'=-\frac{1}{2}$$

(use (8)). This leads to the canonical form

$$\frac{d^2\eta}{d\xi^2}-\xi\frac{d\eta}{d\xi}-\alpha\eta=0,\ddagger \tag{16}$$

$$\alpha=\frac{1}{2}+\frac{1}{2}\sqrt{-I}.$$

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* Crelle's Journal, Vol. 10, p. 92.

† Crelle's Journal, Vol. 17, p. 363; see also Pochhammer, Math. Ann., Vol. 38, pp. 242, 247.

‡ Weiler, l. c., p. 128, and Pochhammer, Math. Ann., Vol. 38, p. 241.

Finally we mention the following special cases:

If $H \neq 0$, $D = 0$, (11) becomes

$$\frac{d^2u}{dz^2} + t^2 v = 0. \tag{17}$$

(Special case of *Riccati's equation.*)

$I = -1$ is the necessary and sufficient condition that (C) shall be reducible to the form

$$\frac{d^2y'}{dx'^2} + 2(a' + b'x')\frac{dy'}{dx'} = 0. \tag{18}$$

UNIVERSITY OF CHICAGO, *March*, 1893.

On Certain Properties of Symmetric, Skew Symmetric, and Orthogonal Matrices.

BY W. H. METZLER.

§1.—INTRODUCTION.

In the Proceedings of the London Mathematical Society (Vol. XXII, Nos. 427, 428), Dr. Henry Taber has proved the following theorem: A real symmetric matrix less one of its multiple latent roots has a nullity equal to its vacuity. He also states and proves that this same property is characteristic of real skew symmetric and of real orthogonal matrices, that the latent roots of a real skew symmetric matrix are either zero or pure imaginary, and that the modulus of every latent root of a real orthogonal matrix is equal to unity. The method employed by Dr. Taber is an extension of that given by A. Buchheim in the Mess. Math. ((2), Vol. XIV). By employing a different method I shall prove the same properties of these matrices.

In addition, I shall give a representation of an orthogonal matrix which admits of both $+1$ and -1 as latent roots, and which at the same time contains a number of arbitrary parameters.

In what follows I shall consider only real matrices, so that when speaking of a matrix it will be understood that a real matrix is meant.

§2.—*Symmetric Matrices.*

It is a well-known theorem that the latent roots of a symmetric matrix are all real. Suppose the symmetric matrix ϕ has as latent roots $g_1, g_2, \ldots . g_s$ occurring $p_1, p_2, \ldots . p_s$ times respectively, then it is also well known that every first minor of the content of ϕ has as roots $g_1, g_2, \ldots . g_s$ occurring $p_1-1, p_2-1, \ldots . p_s-1$ times respectively, and generally every λ^{th} minor of the content of ϕ has as

roots $g_1, g_2, \ldots . g_s$ occurring $p_1 - \lambda$, $p_2 - \lambda$, $\ldots . p_s - \lambda$ times respectively.* The latent roots of the matrix $(\phi - g_1)$ will then, according to the law of latency, be 0, $g_2 - g_1, \ldots . g_s - g_1$ occurring $p_1, p_2, \ldots . p_s$ times respectively; that is, $(\phi - g_1)$ has a vacuity p_1. Obviously every minor of the content of $(\phi - g_1)$ up to the $(p_1 - 1)^{st}$ is vacuous and therefore $(\phi - g_1)$ has a nullity p_1.

§3.—*Skew Symmetric Matrices.*

The latent function of a skew symmetric matrix of even order may be written in the form

$$x^{2n} + \Sigma A_1^2 x^{2n-2} + \Sigma A_2^2 x^{2n-4} + \ldots . + \Sigma A_{n-1}^2 x^2 + A_n^2,$$

where the exponents are all even and the coefficients are all positive; consequently if $A_n^2 \neq 0$, the roots of this function are all imaginary. If $A_n^2 = 0$, $\Sigma A_{n-1}^2 = 0$, $\Sigma A_{n-2}^2 = 0 \ldots . . \Sigma A_{n-\lambda+1}^2 = 0$, but $\Sigma A_{n-\lambda} \neq 0$, then the latent function becomes

$$x^{2\lambda}\{x^{2n-2\lambda} + \Sigma A_1^2 x^{2n-2\lambda-2} + \ldots . + \Sigma A_{n-\lambda}^2\},$$

and therefore 2λ of the roots are zero and the remainder are imaginary. If the order of the matrix is odd, then there is at least one latent root zero since its determinant vanishes, and as in the case of even order, the roots are either zero or imaginary. But the square of a skew symmetric matrix is a symmetric matrix, and since the latent roots of the square of a matrix are the squares of the latent roots of the matrix, we see that the latent roots of the skew symmetric matrix must have been all pure imaginary to have their squares real. Consequently the latent roots of a skew symmetric matrix are either zero or pure imaginary.

Suppose the skew symmetric matrix ϕ has as latent roots

$g_1 i$	occurring	p_1	times,
$-g_1 i$	"	p_1	"
$g_2 i$	"	p_2	"
$-g_2 i$	"	p_2	"
etc.,		etc.	

* Vide Burnside and Panton, Theory of Equations, 2d ed., art. 129, examples 32 and 33.

Then $(\phi - g_1 i)$ is a skew matrix whose latent roots are

$$\begin{array}{rcll} 0 & \text{occurring} & p_1 & \text{times,} \\ -2g_1 i & \text{"} & p_1 & \text{"} \\ (g_2 - g_1)\, i & \text{"} & p_2 & \text{"} \\ -(g_2 + g_1)\, i & \text{"} & p_2 & \text{"} \\ \text{etc.,} & & \text{etc.} & \end{array}$$

Also $(\phi^2 + g_1^2)$ is a symmetric matrix whose latent roots are

$$\begin{array}{rcll} 0 & \text{occurring} & 2p_1 & \text{times,} \\ -g_2^2 + g_1^2 & \text{"} & 2p_2 & \text{"} \\ \text{etc.,} & & \text{etc.} & \end{array}$$

Again, $N_y[\phi - g_1 i] \overline{\gtrless} p_1$, where $N_y[\phi] \overline{\gtrless} k$ means "ϕ has a nullity equal to or less than k"; $N_y[\phi^2 + g_1^2] = 2p_1$, $(\phi^2 + g_1^2)$ being a symmetric matrix, and therefore, as we have shown, has a nullity equal to its vacuity;

$$N_y[\phi + g_1 i] \overline{\gtrless} p_1.$$

But $\phi^2 + g_1^2 = (\phi - g_1 i)(\phi + g_1 i)$,

and therefore $$N_y[\phi - g_1 i] = p_1,$$

and $$N_y[\phi + g_1 i] = p_1,$$

since the product has a nullity $2p_1$ and each of the factors a nullity at most equal to p_1. The vacuity of $(\phi - g_1 i)$ is p_1, and therefore a skew symmetric matrix less one of its latent roots has a nullity equal to its vacuity.

§4.—*Orthogonal Matrices.*

If ϕ is an orthogonal matrix, then the content of ϕ is equal to plus or minus unity and $\phi\breve{\phi} = \breve{\phi}\phi = 1$, where $\breve{\phi}$ is the transverse of ϕ;

$$\therefore\ \breve{\phi} = \phi^{-1}.$$

Let the latent roots of ϕ, which are the same as the latent roots of $\breve{\phi}$, be $g_1, g_2, \ldots g_s$ occurring $p_1, p_2, \ldots . p_s$ times respectively. The roots of ϕ^{-1} will be $g_1^{-1}, g_2^{-1}, \ldots . g_s^{-1}$ occurring $p_1, p_2, \ldots . p_s$ times respectively. But the equa-

tion $\phi = \phi^{-1}$ shows that the latent roots of ϕ^{-1} are the same as the latent roots of $\breve{\phi}$, and consequently the same as the latent roots of ϕ.

Two possible cases present themselves,

$$\begin{aligned} &1).\quad g_\lambda = g_\lambda^{-1} \text{ and } \therefore\ g_\lambda = \pm 1, \\ &2).\quad g_\mu = g_\nu^{-1}. \end{aligned}$$

If then g_1 is a latent root p_1 times, g_1^{-1} will also be a latent root p_1 times.

Since the product of all the latent roots of ϕ is equal to ± 1, we see that if the order of the matrix is odd, at least one of its latent roots must be ± 1. We may therefore take the roots of ϕ to be

$$\begin{array}{lcl} g_1 & \text{occurring } p_1 & \text{times,} \\ g_1^{-1} & \text{"} \quad p_1 & \text{"} \\ g_2 & \text{"} \quad p_2 & \text{"} \\ g_2^{-1} & \text{"} \quad p_2 & \text{"} \\ \text{etc.,} & \text{etc.,} & \end{array}$$

where g_λ may be equal to g_λ^{-1}.*

The latent roots of the symmetric matrix $(\phi + \breve{\phi})$ are

$$\begin{array}{lcl} r_1 = g_1 + g_1^{-1} & \text{occurring} & 2p_1 \text{ times,} \\ r_2 = g_2 + g_2^{-1} & \text{"} & 2p_2 \quad \text{"} \\ r_3 = g_3 + g_3^{-1} & \text{"} & 2p_3 \quad \text{"} \\ \quad \text{etc.,} & & \text{etc.,} \end{array}$$

all of which are real.

The product $(\phi - g_1)(\breve{\phi} - g_1) = \phi\breve{\phi} - g_1(\phi + \breve{\phi}) + g_1^2 = 1 + g_1^2 - g_1(\phi + \breve{\phi})$ is a symmetric matrix whose latent roots are

$$\begin{array}{rcl} 0 & \text{occurring} & 2p_1 \text{ times,} \\ 1 + g_1^2 - g_1(g_2 + g_2^{-1}) & \text{"} & 2p_2 \quad \text{"} \\ \text{etc.,} & & \text{etc.} \end{array}$$

*** That the orthogonal matrix ϕ is symmetric when its latent roots are all real (i. e. equal to ± 1) may be shown as follows:**

The matrix $(\phi - \breve{\phi})$ is skew symmetric, having as latent roots $\frac{g_1^2 - 1}{g_1}$, $\frac{g_2^2 - 1}{g_2}$, etc., occurring $2p_1$, $2p_2$, etc., times respectively. But if the latent roots of ϕ are real, then all the latent roots of $(\phi - \breve{\phi})$ are zero. The sum of the products of the latent roots, two at a time, is equal to the sum of the principal minors of order two of the content of ϕ, and consequently we have $\Sigma(\phi_{rs} - \phi_{sr})^2 = 0$, where ϕ_{rs} is the constituent of ϕ in the r^{th} row and s^{th} column,

$$\therefore\ \phi_{rs} = \phi_{sr}:$$

that is, ϕ is symmetric.

Again,
$$N_v[\phi - g_1] \overline{\overline{<}} p_1,$$
$$N_v[\breve{\phi} - g_1] \overline{\overline{<}} p_1;$$
but
$$N_v[(\phi - g_1)(\breve{\phi} - g_1)] = 2p_1;^*$$
$$\therefore N_v[\phi - g_1] = N_v[\breve{\phi} - g_1] = p_1.$$

The vacuity of $(\phi - g_1)$ is p_1, and therefore an orthogonal matrix less one of its latent roots has a nullity equal to its vacuity.

The matrix $(\phi - \breve{\phi})$ is skew symmetric, having as latent roots $\frac{g_1^2 - 1}{g_1}$, $\frac{g_2^2 - 1}{g_2}$, etc., which are either zero or pure imaginary.

We have then
$$\frac{g_1^2 + 1}{g_1} = r_1,$$
$$\frac{g_1^2 - 1}{g_1} = h_1 i,$$
where h_1 is real, therefore
$$\frac{r_1}{2} + \frac{h_1}{2} i = g_1$$
and
$$\frac{r_1}{2} - \frac{h_1}{2} i = \frac{1}{g_1},$$
$\therefore \left(\frac{r_1}{2}\right)^2 + \left(\frac{h_1}{2}\right)^2 = 1$; that is, the modulus of g_1 is unity, and similarly the modulus of each of the g's is unity.

The Representation of an Orthogonal Matrix.

The representation of an orthogonal matrix in terms of $\frac{n(n-1)}{2}$ arbitrary quantities was given by M. Hermit as follows:
$$\phi = \left(\begin{array}{|cccc|} 2\frac{\beta_{11} - \Delta}{\Delta} & 2\frac{\beta_{12}}{\Delta} & 2\frac{\beta_{13}}{\Delta} & \cdots\cdots, \\ 2\frac{\beta_{21}}{\Delta} & 2\frac{\beta_{22} - \Delta}{\Delta} & 2\frac{\beta_{23}}{\Delta} & \cdots\cdots, \\ 2\frac{\beta_{31}}{\Delta} & 2\frac{\beta_{32}}{\Delta} & 2\frac{\beta_{33} - \Delta}{\Delta} & \cdots\cdots, \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right)^\dagger$$

---

*The matrix $1 + g_1^2 - g_1(\phi + \breve{\phi}) = g_1\{r_1 - (\phi + \breve{\phi})\} = \psi$ though symmetric, is not a real matrix unless g_1 is real. It is, however, obviously true that ψ less either of its latent roots has a nullity equal to its vacuity.

†Camb. and Dub. Math. Jour., Vol. IX (1858), p. 63; vide Salmon, Higher Algebra, art. 44.

where β_{11}, β_{12}, etc., are the minors of the arbitrary skew determinant

$$\Delta = \begin{vmatrix} 1 & b_{12} & b_{13} \dots, \\ -b_{12} & 1 & b_{23} \dots, \\ -b_{13} & -b_{23} & 1 \dots, \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

If ψ denotes the skew symmetric matrix

$$\begin{pmatrix} 0 & b_{12} & b_{13} \dots, \\ -b_{12} & 0 & b_{23} \dots, \\ -b_{13} & -b_{23} & 0 \dots, \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

then we easily find that

$$\phi = 2/\Delta\{\psi + \psi^2\} + 1 \text{ if } \phi \text{ is of } 3^{d} \text{ order,}$$
$$\phi = 2/\Delta\{\psi + \psi^2 + \psi^3 + \psi(\Delta - 1 - d^2) - d^2\} + 1, \text{ where}$$
$$d^2 = (b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23})^2 \text{ if } \phi \text{ is of } 4^{th} \text{ order,}$$
etc.

Another representation of an orthogonal matrix in terms of an arbitrary skew symmetric matrix is the following:

$$\phi = \frac{1 + \psi}{1 - \psi}.^*$$

I shall now show that these two representations are virtually the same. The matrix

$$(1 - \psi) = \begin{pmatrix} 1 & -b_{12} & -b_{13} \dots, \\ b_{12} & 1 & -b_{23} \dots, \\ b_{13} & b_{23} & 1 \dots, \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

$$\therefore (1 - \psi)^{-1} = 1/\Delta \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \dots, \\ \beta_{21} & \beta_{22} & \beta_{23} \dots, \\ \beta_{31} & \beta_{32} & \beta_{33} \dots, \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

---

* Prof. Cayley, Philosophical Trans., 1858.

and

$$\therefore\ \phi=(1+\psi)(1-\psi)^{-1}=1/\Delta\begin{pmatrix} 1 & b_{12} & b_{13}\ldots, \\ -b_{12} & 1 & b_{23}\ldots, \\ -b_{13} & -b_{23} & 1\ \ldots, \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13}\ldots, \\ \beta_{21} & \beta_{22} & \beta_{23}\ldots, \\ \beta_{31} & \beta_{32} & \beta_{33}\ldots, \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

$$=1/\Delta\begin{pmatrix} \beta_{11}+b_{12}\beta_{21}+b_{13}\beta_{31} & \beta_{12}+b_{12}\beta_{22}+b_{13}\beta_{32} & \beta_{13}+b_{12}\beta_{23}+b_{13}\beta_{33}\ldots, \\ -b_{12}\beta_{11}+\beta_{21}+b_{23}\beta_{31} & -b_{12}\beta_{12}+\beta_{22}+b_{23}\beta_{32} & -b_{12}\beta_{13}+\beta_{23}+b_{23}\beta_{33}\ldots, \\ -b_{13}\beta_{11}-b_{23}\beta_{21}+\beta_{31} & -b_{13}\beta_{12}-b_{23}\beta_{22}+\beta_{32} & -b_{13}\beta_{13}-b_{23}\beta_{23}+\beta_{33}\ldots, \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

$$=\begin{pmatrix} 2\dfrac{\beta_{11}-\Delta}{\Delta} & 2\dfrac{\beta_{12}}{\Delta} & 2\dfrac{\beta_{13}}{\Delta} & \ldots, \\ 2\dfrac{\beta_{21}}{\Delta} & 2\dfrac{\beta_{22}-\Delta}{\Delta} & 2\dfrac{\beta_{23}}{\Delta} & \ldots, \\ 2\dfrac{\beta_{31}}{\Delta} & 2\dfrac{\beta_{32}}{\Delta} & 2\dfrac{\beta_{33}-\Delta}{\Delta} & \ldots, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

These representations fail for the case of an orthogonal matrix having both $+1$ and -1 as latent roots.

If we take

$$\phi_1=\theta\phi=\theta/\Delta\begin{pmatrix} 2\beta_{11}-\Delta & 2\beta_{12} & 2\beta_{13} & \ldots, \\ 2\beta_{21} & 2\beta_{22}-\Delta & 2\beta_{23} & \ldots, \\ 2\beta_{31} & 2\beta_{32} & 2\beta_{33}-\Delta & \ldots, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

where $\theta=\begin{pmatrix} b_{11} & 0 & 0 & \ldots, \\ 0 & b_{22} & 0 & \ldots, \\ 0 & 0 & b_{33} & \ldots, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$ and $b_{11}=\pm 1$, $b_{22}=\pm 1$, $b_{33}=\pm 1$, etc.

we will obviously have a matrix satisfying the conditions of orthogonality, and which includes ϕ as a special case.

Taking as an example a matrix of the 3^d order, we have

$$
\begin{aligned}
|\phi_1+1| = b_{11}b_{22}b_{33} + \frac{1}{\Delta^2}\,[&b_{22}b_{33}\{(2\beta_{33}-\Delta)(2\beta_{22}-\Delta)-4\beta_{23}\beta_{32}\} \\
&+ b_{11}b_{22}\{(2\beta_{11}-\Delta)(2\beta_{22}-\Delta)-4\beta_{12}\beta_{21}\} \\
&+ b_{11}b_{33}\{(2\beta_{11}-\Delta)(2\beta_{33}-\Delta)-4\beta_{13}\beta_{31}\}] \\
+\frac{1}{\Delta}\,&[b_{33}(2\beta_{33}-\Delta)+b_{22}(2\beta_{22}-\Delta)+b_{11}(2\beta_{11}-\Delta)]+1 \\
= b_{11}b_{22}b_{33} + \frac{1}{\Delta^2}\,[&(b_{11}b_{22}+b_{11}b_{33}+b_{22}b_{33})(4\Delta+\Delta^2)-2\Delta\{b_{22}b_{33}(\beta_{22}+\beta_{33}) \\
&+ b_{11}b_{22}(\beta_{11}+\beta_{22})+b_{11}b_{33}(\beta_{11}+\beta_{33})\}] \\
+\frac{1}{\Delta}\,&[2(b_{11}\beta_{11}+b_{22}\beta_{22}+b_{33}\beta_{33})-\Delta(b\quad+b_{22}+b_{33})]+1,
\end{aligned}
$$

and

$$
\begin{aligned}
|\phi_1-1| = b_{11}b_{22}b_{33} - \frac{1}{\Delta^2}\,[&(b_{11}b_{22}+b_{11}b_{33}+b_{22}b_{33})(4\Delta+\Delta^2)-2\Delta\{b_{22}b_{33}(\beta_{22}+\beta_{33} \\
&+ b_{11}b_{22}(\beta_{11}+\beta_{22})+b_{11}b_{33}(\beta_{11}+\beta_{33})\}] \\
+\frac{1}{\Delta}\,&[2(b_{11}\beta_{11}+b_{22}\beta_{22}+b_{33}\beta_{33})-\Delta(b_{11}+b_{22}+b_{33})]-1.
\end{aligned}
$$

These results give

$$
\begin{aligned}
|\phi_1+1| &= 0 \text{ for } b_{11}=b_{22}=b_{33}=-1 \text{ or } b_{11}=-1 \text{ and } b_{22}=b_{33}=\ \ 1,^* \\
|\phi_1-1| &= 0 \quad\text{“}\quad b_{11}=b_{22}=b_{33}=\ \ 1 \quad\text{“}\quad b_{11}=\ \ 1 \quad\text{“}\quad b_{22}=b_{33}=-1,^*
\end{aligned}
$$

$$
\begin{aligned}
|\phi_1+1| &= \frac{8b_{12}^2}{\Delta} \text{ for } b_{11}=b_{22}=-1 \text{ and } b_{33}=\ \ 1, \\
&= \frac{8b_{13}^2}{\Delta} \quad\text{“}\quad b_{11}=b_{33}=-1 \text{ and } b_{22}=\ \ 1, \\
&= \frac{8b_{23}^2}{\Delta} \quad\text{“}\quad b_{22}=b_{33}=-1 \text{ and } b_{11}=\ \ 1,
\end{aligned}
$$

---

*It is obvious that $|\phi_1+1|=0$ for $b_{22}=-1$ and $b_{11}=b_{33}=1$, and for $b_{33}=-1$ and $b_{11}=b_{22}=1$, as well as for the case given. Similarly in case of $|\phi_1-1|$.

and

$$|\phi_1 - 1| = -\frac{8b_{12}^2}{\Delta} \text{ for } b_{11} = b_{22} = \quad 1 \text{ and } b_{33} = -1,$$

$$= -\frac{8b_{13}^2}{\Delta} \text{ " } b_{11} = b_{33} = \quad 1 \text{ and } b_{22} = -1,$$

$$= -\frac{8b_{23}^2}{\Delta} \text{ " } b_{22} = b_{33} = \quad 1 \text{ and } b_{11} = -1.$$

The matrix ϕ_1 is therefore such that both $+1$ and -1 are latent roots, provided one of the constituents of ψ vanish.

Similarly for matrices of higher order.

CLARK UNIVERSITY, *June* 1, 1892.

A Deduction and Demonstration of Taylor's Formula.

By W. H. Echols.

The following method of deducing the formula for the expansion of $f(x+h)$ in terms of ascending powers of h is of interest, because it does not require the assumption of the possibility of the series nor that it should be differentiable.

The determinant

$$\begin{vmatrix} fx\ , & 1, & x\ , & \ldots\ldots, & x^n \\ fa_0, & 1, & a_0, & \ldots\ldots, & a_0^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ fa_n, & 1, & a_n, & \ldots\ldots, & a_n^n \end{vmatrix} \tag{1}$$

vanishes for the $n+1$ values of x, $a_0 \ldots\ldots a_n$. Its first derivative vanishes for n values of x between these values, by Rolle's theorem, fx being a continuous function for the limits prescribed. Its second derivative vanishes for $n-1$ values of x between these values, and so on, until evidently its n^{th} derivative vanishes for some value u, of x, which lies between the greatest and least of the values $a_0 \ldots\ldots a_n$.

Let Fx represent the above determinant, and for brevity write

$$\zeta^1 = \zeta^1(a_0 \ldots\ldots a_n).$$

We then have

$$Fx = \zeta^1 fx + \phi x, \tag{2}$$

wherein ϕx is a rational integral function of the n^{th} degree.

This being so, we have

$$\begin{aligned} F(x+h) &= \zeta^1 f(x+h) + \phi(x+h) \\ &= \zeta^1 f(x+h) + \phi x + \frac{h^1}{1!}\phi' x + \ldots\ldots + \frac{h^n}{n!}\phi^n x. \end{aligned} \tag{3}$$

Differentiate (2) n times and multiply these equations through respectively by $h^r/r!$, $(r=1 \ldots . n)$, whence results

$$hFx = \zeta^k hf'x + h\phi'x,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{h^{n-1}}{(n-1)!}F^{n-1}x = \zeta^k\frac{h^{n-1}}{(n-1)!}f^{n-1}x + \frac{h^{n-1}}{(n-1)!}\phi^{n-1}x,$$

$$\frac{h^n}{n!}F^n u = \zeta^k\frac{h^n}{n!}f^n u + \frac{h^n}{n!}\phi^n u = 0.$$

Subtracting these equations from (3), member by member, we obtain

$$f(x+h) - fx - \frac{h^1}{1!}f'x - \ldots . - \frac{h^{n-1}}{(n-1)!}f^{n-1}x - \frac{h^n}{n!}f^n u$$

$$= \frac{1}{\zeta^k}\left[F(x+h) - Fx - \frac{h^1}{1!}F'x - \ldots . - \frac{h^{n-1}}{(n-1)!}F^{n-1}x\right]. \quad (4)$$

Since the a's are arbitrary, we may shift them as we choose, so put $a_0 = x$ and $a_n = x + h$, then $Fx = 0$, also $F(x+h) = 0$, and the second member of (4) becomes

$$-\frac{\frac{h^1}{1!}F'x + \ldots . + \frac{h^{n-1}}{(n-1)!}F^{n-1}x}{\zeta^k(x,\ a_1 \ldots . a_{n-1},\ x+h)},$$

which takes the indeterminate form $0/0$ whenever $a_1, \ldots ., a_{n-1} = x$.

To evaluate the true form of this ratio when $a_1, \ldots ., a_{n-1} = x$, apply to the numerator and denominator the operator

$$\left(\frac{d}{da_1}\right)^1_{a_1=x} \ldots . \left(\frac{d}{da_{n-1}}\right)^{n-1}_{a_{n-1}=x},$$

$\left(\frac{d}{da_r}\right)^r_{a_r=x}$ causes $F^r x$ to vanish $(r = 1 \ldots . n-1)$, while

$$\left(\frac{d}{da_1}\right)^1_{a_1=x} \ldots . \left(\frac{d}{da_{n-1}}\right)^{n-1}_{a_{n-1}=x} \zeta^k(x,\ a_0, \ldots ., a_{n-1},\ x+h) = (n-1)!!\, h^n.$$

Hence the true value of the ratio is zero and we have

$$f(x+h) = fx + \frac{h^1}{1!}f'x + \ldots . + \frac{h^{n-1}}{(n-1)!}f^{n-1}x + \frac{h^n}{n!}f^n u. \quad (x < u < x+h)$$

The method of determining the ultimate ratio of an indeterminate form can be developed wholly independent of Taylor's formula (Todhunter's Diff. Calculus, p. 124); it seems, therefore, that the above analysis is free from objection.

Announcement by the Physico-Mathematical Society of the University of Kasan.

The 22d of October, 1893, will be the one-hundredth anniversary of the birth of the famous Russian mathematician, Lobatcheffsky. This "Copernicus of geometry," as Clifford has called him, belongs undoubtedly to that class of investigators who have not only enriched science by the contribution of important facts, but have revolutionized the very fundamental notions of science which their predecessors considered unassailable.

From the time of Euclid no one had doubted the truth of his axioms and postulates; all the efforts of mathematicians in this direction were toward the limiting of these to the fewest number possible. A great number of attempts, for example, were made to derive the last axiom from the others, while its truth was never questioned. Lobatcheffsky was the first to perceive that the question was one to be decided by experiment; he showed clearly that the assumption of this axiom is equivalent to endowing our space with certain qualities which it may or may not have; and finally, he showed the possibility of a more general geometry without making use of this axiom. Although these ideas gained ground slowly, many of the most distinguished geometers of recent times have testified to the great value of Lobatcheffsky's work, and have shown that his geometry of two dimensions is the geometry of a surface of constant negative curvature, while the geometry of three dimensions introduces the new ideas of hyper-space and the curvature of ordinary space.

The scientific value of Lobatcheffsky's researches is scarcely greater than their philosophical importance. On the one hand, they conduct us to a new question as to the properties of space; on the other hand, they throw a new light upon the question of the origin of our geometrical axioms, and for that reason have a great importance in the theory of perception.

It was the good fortune of the Imperial University of Kasan to count Lobatcheffsky as one of its pupils and members. Here he fulfilled the duties of

a professor from 1812 to 1846, and those of Rector from 1827 to 1846. He is dear to this institution, not only on account of his scientific attainments, but also because of his activity as an instructor. The history of his life and works is inseparably bound up with the history of the University of Kasan; it owes to him the construction of its best buildings and the organization of its library.

The Physico-Mathematical Society of the Imperial University of Kasan cannot neglect to call attention to the approaching centennial anniversary of the birth of the great Russian geometer.

With Imperial sanction, the Physico-Mathematical Society addresses itself to the friends of science in all countries, asking them to contribute to a fund which shall bear the name of Lobatcheffsky. According to the contributions, the Society proposes to devote this fund, either to the establishment of a prize for mathematical researches, or to the erection of a bust in the buildings of the University. It is hoped that the subscriptions will be sufficient to accomplish both of these objects.

Subscriptions should be addressed to The Physico-Mathematical Society, Kasan, Russia.

A. Wassilieff, *President.*
T. Souvoroff, *Vice-President.*

On Toroidal Functions.

By A. B. Basset, M. A., F. R. S.

1. Every spherical surface harmonic of degree n can be expressed in the form of the series

$$\Sigma_{m=0}^{m=n} A_m P_n^m(\nu)\cos(m\phi + \alpha_m),$$

where P_n^m is an associated function of the *first* kind of degree n and order m. This function satisfies the differential equation

$$\frac{d}{d\nu}(1-\nu^2)\frac{du}{d\nu} - \frac{m^2 u}{1-\nu^2} + n(n+1)u = 0. \qquad (1)$$

In the theory of spherical harmonics $\nu = \cos\theta$, and therefore lies between the limits 1 and -1; but in developing the theory of associated functions there is no necessity whatever to impose this limitation on the value of ν. In fact, in the theory of the potentials of ovary ellipsoids, associated functions occur in which the argument is never less than unity,* and as one of my objects is to develop the theory of toroidal functions from a point of view which brings out their connection with ordinary associated functions, the argument will always be supposed to be not less than unity. Many of the results obtained will be found to be universally true for all real values of the argument, whilst other results, when the argument lies between 1 and -1, can easily be deduced therefrom.

It is also known that if u_0 be any solution of the equation to which (1) reduces when $m = 0$, a solution of (1) is

$$u = (\nu^2 - 1)^{\frac{1}{2}m}\frac{d^m u_0}{d\nu^m}.$$

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*Prof. Hicks describes these functions (and also toroidal functions) as spherical harmonics of imaginary argument. See Phil. Trans., 1881, pp. 613 and 617. In order to understand this statement it must be borne in mind that he takes the argument to be θ, where

$$\nu = \cosh\theta = \cos\iota\theta.$$

The two kinds of associated functions are therefore defined by the equations

$$\left.\begin{aligned} P_n^m &= (\nu^2-1)^{\frac{1}{2}m}\frac{d^mP_n}{d\nu^m}, \\ Q_n^m &= (\nu^2-1)^{\frac{1}{2}m}\frac{d^mQ_n}{d\nu^m}, \end{aligned}\right\} \tag{2}$$

where P_n, Q_n are the zonal harmonics of the first and second kinds respectively. Accordingly Q_n^m is an associated function of the *second* kind of degree n and order m. In ordinary spherical harmonic analysis the factor must be changed into $(1-\nu^2)^{\frac{1}{2}m}$ in the first equation in order to avoid the unnecessary* introduction of imaginary quantities. The corresponding change in the function Q_n^m need not be considered, because it never occurs in physical investigations in which ν lies between 1 and -1.

It was shown by Laplace that P_n can be expressed in the form of the definite integral

$$P_n = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cos\phi\}^{n+1}}, \tag{3}$$

and it has been shown by Heine and myself† that Q_n can be expressed in the form

$$Q_n = \int_0^\infty \frac{d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cosh\phi\}^{n+1}}. \tag{4}$$

From these equations combined with (2) it follows that P_n^m and Q_n^m can also be expressed in the forms of definite integrals, in which case we shall have‡

$$P_n^m = \frac{(-)^m n!}{(n-m)!}\int_0^\pi \frac{\cos m\phi d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cos\phi\}^{n+1}}, \tag{5}$$

$$Q_n^m = \frac{(-)^m n!}{(n-m)!}\int_0^\infty \frac{\cosh m\phi d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cosh\phi\}^{n+1}}. \tag{6}$$

The advantages of these definite integrals are that they furnish concise expressions for associated functions by means of which many of their properties may be easily investigated, and a variety of difference and mixed difference equations obtained connecting functions of different orders and degrees.

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* Those who have had occasion to employ Bessel's functions in any investigation of considerable length can hardly fail to have appreciated the superiority of the notation $I_m(x)$, $K_m(x)$ in the place of $J_m(\iota x)$, $Y_m(\iota x)$. See Hydrodynamics, Chap. XII, and Brit. Assoc. Rep., 1889, p. 28.

† Mess. Math., Vol. XIII, p. 147.

‡ Ibid., pp. 150, 152.

2. A toroidal function is an associated function of degree $n-\frac{1}{2}$ and order m; and the notation which ought in strictness to be adopted for the two kinds of toroidal functions is $P^m_{n-\frac{1}{2}}$ and $Q^m_{n-\frac{1}{2}}$; but as these functions rarely if ever occur in an investigation which also involves associated functions of integral degree n, it will be generally sufficient to employ the suffix n instead of $n-\frac{1}{2}$.

Under these circumstances we should anticipate that the expressions for the two toroidal functions in terms of definite integrals would be derived from (5) and (6) by changing n into $n-\frac{1}{2}$, so that we may write

$$P^m_n = \frac{(-)^m\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}\int_0^\pi \frac{\cos m\phi d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cos\phi\}^{n+\frac{1}{2}}}, \tag{7}$$

$$Q^m_n = \frac{(-)^m\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}\int_0^\infty \frac{\cosh m\phi d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cosh\phi\}^{n+\frac{1}{2}}}, \tag{8}$$

and we shall presently show that this is the case.

The theory of toroidal functions was first investigated by Prof. W. M. Hicks as a means of discussing the motion of circular vortex rings. He has, however, (as frequently happens when a new branch of mathematics is being investigated for the first time), presented the subject in a somewhat complicated form. I therefore propose in the present paper to develop the subject by means of the two definite integrals (7) and (8), to correct some errors which Prof. Hicks has made, and also to extend his results.

3. Putting $n-\frac{1}{2}$ for n in (1), the differential equation for toroidal functions becomes

$$\frac{d}{d\nu}(1-\nu^2)\frac{du}{d\nu} - \frac{m^2u}{1-\nu^2} + (n^2-\tfrac{1}{4})u = 0, \tag{9}$$

where n is zero or any positive integer whatever, and m is zero or any positive integer which is not greater than n.

Let A^m_n denote the coefficient of the definite integral in (7); also let $D = \nu + (\nu^2-1)^{\frac{1}{2}}\cos\phi$, then from (7) we get

$$(\nu^2-1)\frac{dP^m_n}{d\nu} = -(n+\tfrac{1}{2})A^m_n\int_0^\pi \frac{\nu^2-1+\nu(\nu^2-1)\cos\phi}{D^{n+\frac{3}{2}}}\cos m\phi d\phi. \tag{10}$$

Now

$$A^m_{n+1} = \frac{n+\frac{1}{2}}{n-m+\frac{1}{2}}A^m_n,$$

whence

$$(\nu^2-1)\frac{dP^m_n}{d\nu} = -(n+\tfrac{1}{2})\left(\nu P^m_n - \frac{n-m+\frac{1}{2}}{n+\frac{1}{2}}P^m_{n+1}\right). \tag{11}$$

Again, from (10) we obtain

$$(\nu^2-1)\frac{dP_n^m}{d\nu} = -(n+\tfrac{1}{2})(\nu^2-1)^{\frac{1}{2}}A_n^m\left\{\int_0^\pi \frac{\cos\phi\cos m\phi d\phi}{D^{n+\frac{1}{2}}} + (\nu^2-1)^{\frac{1}{2}}\int_0^\pi \frac{\sin^2\phi\cos m\phi d\phi}{D^{n+\frac{3}{2}}}\right\}.$$

Integrating the last term by parts we obtain

$$(\nu^2-1)\frac{dP_n^m}{d\nu} = -(\nu^2-1)^{\frac{1}{2}}A_n^m\left\{(n-\tfrac{1}{2})\int_0^\pi \frac{\cos\phi\cos m\phi d\phi}{D^{n+\frac{1}{2}}} + m\int_0^\pi \frac{\sin\phi\sin m\phi d\phi}{D^{n+\frac{1}{2}}}\right\}. \quad (12)$$

Integrating the last term of (12) by parts we finally obtain

$$(\nu^2-1)\frac{dP_n^m}{d\nu} = -(n-\tfrac{1}{2})\left(\frac{n-\frac{1}{2}}{n-m-\frac{1}{2}}P_{n-1}^m - \nu P_n^m\right) + \frac{m^2P_{n-1}^m}{n-m-\frac{1}{2}}. \quad (13)$$

Again, from (11) we get

$$\frac{d}{d\nu}(1-\nu^2)\frac{dP_n^m}{d\nu} = (n+\tfrac{1}{2})\left(P_n^m + \nu\frac{dP_n^m}{d\nu} - \frac{n-m+\frac{1}{2}}{n+\frac{1}{2}}\frac{dP_{n+1}^m}{d\nu}\right).$$

Substituting the value of $dP_n^m/d\nu$ from (11), and that of $dP_{n+1}^m/d\nu$ from (13), the right-hand side becomes

$$-(n^2-\tfrac{1}{4})P_n^m + \frac{m^2P_n^m}{1-\nu^2},$$

which shows that the function P_n^m as defined by (7) satisfies the differential equation (9).

Eliminating $dP_n^m/d\nu$ between (11) and (13), we obtain the sequence equation

$$(n-m+\tfrac{1}{2})P_{n+1}^m - 2n\nu P_n^m + \frac{(n-\frac{1}{2})^2}{(n-m-\frac{1}{2})}P_{n-1}^m = \frac{m^2P_{n-1}^m}{n-m-\frac{1}{2}}. \quad (14)$$

Equations (11), (13) and (14) are analogous to equations (55), (54) and (56) of §273 of my Hydrodynamics, to which they reduce when $m=0$. In fact the whole investigation is on all fours with that section, excepting that it is more general since m is not supposed to be zero; and the simplification which is obtained by first considering the case of m zero and afterwards proceeding to the more general case in which m is a positive integer, is so very slight that it is better to commence with the general case.

4. Prof. Hicks has obtained equations corresponding to (11), (13) and (14), viz. equations (26) and (27) on page 631 of his paper, which I believe are erroneous. His notation is somewhat different from mine, and I shall therefore explain it in order that the reader may be able to examine the question. He writes

$$C = \cosh u = \nu,$$
$$S = \sinh u = (\nu^2 - 1)^{\frac{1}{2}},$$

and on page 636 he defines the toroidal function of the first kind by the equation

$$P_{m,n} = \int_0^\pi \frac{\cos m\theta d\theta}{(C - S\cos\theta)^{n+\frac{1}{2}}},$$

from which it follows that

$$P_n^m = -A_n^m P_{m,n}.$$

Also the accents denote differentiation with respect to u, so that

$$A_n^m P'_{m,n} = -S\frac{dP_n^m}{d\nu}.$$

It therefore follows that equations (26) of his paper ought to be

$$2SP'_{m,n} = (2n+1)(P_{m,n+1} - CP_{m,n}),$$
$$2SP'_{m,n} = (2n-1)(CP_{m,n} - P_{m,n-1}) + \frac{4m^2 P_{m,n-1}}{2n-1},$$

and consequently his sequence equation (27) is also wrong.

5. We shall now show that the expression for P_n^m satisfies (2).

If P_n^m be any function of ν which satisfies (2), it follows at once that

$$P_n^{m+1} = (\nu^2 - 1)^{\frac{1}{2}}\frac{dP_n^m}{d\nu} - \frac{m\nu}{(\nu^2-1)^{\frac{1}{2}}}P_n^m, \tag{15}$$

and we have to show that if the given value of P_n^m is substituted in the right-hand side, the result is equal to P_n^{m+1}.

Assuming the given value of P_n^m, the value of the first term in (15) is determined by (12). The second term gives

$$\frac{m\nu}{A_n^m(\nu^2-1)^{\frac{1}{2}}}P_n^m = \frac{m\nu}{(\nu^2-1)^{\frac{1}{2}}}\int_0^\pi \frac{\cos m\phi d\phi}{D^{n+\frac{1}{2}}}.$$

Integrating by parts, the right-hand side is equal to

$$-(n+\tfrac{1}{2})\,\nu\int_0^\pi \frac{\sin\phi\sin m\phi d\phi}{D^{n+\frac{3}{2}}}$$
$$= -(n+\tfrac{1}{2})\int_0^\pi \frac{\sin\phi\,\sin m\phi d\phi}{D^{n+\frac{1}{2}}} + (n+\tfrac{1}{2})(\nu^2-1)^{\frac{1}{2}}\int_0^\pi \frac{\sin\phi\,\cos\phi\,\sin m\phi d\phi}{D^{n+\frac{3}{2}}}.$$

Integrating the second term on the right-hand side by parts, it becomes

$$-\int_0^\pi \frac{m\cos\phi\cos m\phi - \sin\phi\sin m\phi}{D^{n+\frac{1}{2}}}\,d\phi,$$

whence

$$\frac{m\nu}{A_n^m(\nu^2-1)^{\frac{1}{2}}}P_n^m = -(n-\tfrac{1}{2})\int_0^\pi \frac{\sin\phi\,\sin m\phi d\phi}{D^{n+\frac{1}{2}}} - m\int_0^\pi \frac{\cos\phi\,\cos m\phi d\phi}{D^{n+\frac{1}{2}}}.$$

Accordingly by (12) the right-hand side of (15) becomes

$$-A_n^m(n-m-\tfrac{1}{2})\int_0^\pi \frac{\cos(m+1)\phi}{D^{n+\frac{1}{2}}}\,d\phi.$$

From the value of A_n^m it follows that

$$-A_n^m(n-m-\tfrac{1}{2}) = A_n^{m+1},$$

and therefore the right-hand side of (15) is equal to P_n^{m+1}, which shows that the expression (7) satisfies (2). We have therefore shown that the definite integral (7) satisfies all the necessary conditions.

6. Equations (11) and (13) furnish two equations which connect the differential coefficient of a function of order m and degree n with the functions of order m and degrees $n+1$, n and $n-1$, from which results the sequence equation (14) connecting three functions of order m and degrees $n+1$, n and $n-1$. We shall now establish three similar equations connecting functions of degree n and orders $m+1$, m and $m-1$. The first equation of the latter class is (15); to obtain a second equation we observe that (12) may be written in the form

$$(\nu^2-1)^{\frac{1}{2}}\frac{dP_n^m}{d\nu} = -A_n^m\left\{\frac{n-m-\frac{1}{2}}{2A_n^{m+1}}P_n^{m+1} + \frac{n+m-\frac{1}{2}}{2A_n^{m-1}}P_n^{m-1}\right\}$$
$$= \tfrac{1}{2}P_n^{m+1} + \tfrac{1}{2}(n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})P_n^{m-1}. \quad (16)$$

Eliminating $dP_n^m/d\nu$ between (15) and (16), we get

$$P_n^{m+1} + \frac{2m\nu}{(\nu^2-1)^{\frac{1}{2}}}P_n^m = (n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})P_n^{m-1}. \quad (17)$$

By means of (17), equation (15) may be written

$$(\nu^2-1)\frac{dP_n^m}{d\nu}=(n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})(\nu^2-1)^{\frac{1}{2}}P_n^{m-1}-m\nu P_n^m, \tag{18}$$

whilst (15) is

$$(\nu^2-1)\frac{dP_n^m}{d\nu}=(\nu^2-1)^{\frac{1}{2}}P_n^{m+1}+m\nu P_n^m. \tag{19}$$

7. Equations (17), (18) and (19) connect three toroidal functions of degree n and orders $m+1$, m and $m-1$, and they are the analogues of (14), (13) and (11). The corresponding equations given by Prof. Hicks are (32) and (31) on p. 633 of his paper, and we shall now show that his equations are wrong.

From (18), the equation corresponding to the second of his equations (31) ought to be

$$2SP'_{m,\,n}=-2mCP_{m,\,n}-(2n+2m-1)\,SP_{m-1,\,n},$$

whilst that corresponding to his first equation is

$$2SP'_{m,\,n}=2mCP_{m,\,n}-(2n-2m-1)\,SP_{m+1,\,n},$$

which shows that his sequence equation (32) is also wrong.

8. In the theory of toroidal functions, n is any positive integer, and m is any positive integer which is not greater than n, the value zero being included. Now, if in §§ 3 to 6 we write $n+\frac{1}{2}$ for n, the whole of the preceding analysis will apply to associated functions of the first kind of degree n and order m. The definite integral expression (7) becomes equal to (5), and the three sequence equations (11), (13) and (14), connecting associated functions of order m and degrees $n+1$, n and $n-1$, become

$$\left.\begin{aligned}
(\nu^2-1)\frac{dP_n^m}{d\nu}&=-(n+1)\left(\nu P_n^m-\frac{n-m+1}{n+1}P_{n+1}^m\right),\\
(\nu^2-1)\frac{dP_n^m}{d\nu}&=-n\left(\frac{n}{n-m}P_{n-1}^m-\nu P_n^m\right)+\frac{m^2P_{n-1}^m}{n-m},\\
(n-m+1)P_{n+1}^m-(2n+1)\nu P_n^m+\frac{n^2}{n-m}P_{n-1}^m&=\frac{m^2P_{n-1}^m}{n-m},
\end{aligned}\right\} \tag{20}$$

whilst the three equations (19), (18) and (17) connecting the associated functions degree n and orders $m+1$, m and $m-1$, become

$$\left.\begin{aligned}(\nu^2-1)\frac{dP_n^m}{d\nu} &= (\nu^2-1)\,P_n^{m+1} + m\nu P_n^m,\\ (\nu^2-1)\frac{dP_n^m}{d\nu} &= (n+m)(n-m+1)(\nu^2-1)^{\frac{1}{2}}P_n^{m-1} - m\nu P_n^m,\\ P_n^{m+1} + \frac{2m\nu}{(\nu^2-1)^{\frac{1}{2}}}P_n^m &= (n+m)(n-m+1)\,P_n^{m-1}.\end{aligned}\right\} \tag{21}$$

These formulæ can be proved directly from equation (5).

9. The definite integral expression (8) for toroidal functions of the second kind can be shown by a precisely similar method to satisfy the same equations, but in deducing equations such as (13) or (14) it must be recollected that m must be supposed to be not greater than $n-1$. Similar observations apply to the ordinary associated functions of the second kind, which can be deduced by writing $n+\frac{1}{2}$ for n.

There is one result which is of considerable importance in physical investigations, viz. the value of the quantity

$$P_n'^m Q_n^m - Q_n'^m P_n^m,$$

where the accents denote differentiation with respect to ν. Calling this u_n^m, substitute the values of $P_n'^m$ and $Q_n'^m$ from (18) and we get

$$u_n^m = \frac{(n+m-\frac{1}{2})(n-m+\frac{1}{2})}{(\nu^2-1)^{\frac{1}{2}}}\,(P_n^{m-1}Q_n^m - P_n^m Q_n^{m-1}).$$

In (19) write $m-1$ for m, and then substitute in the last equation the values of Q_n^m and P_n^m and we get

$$u_n^m = -\,(n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})\,u_n^{m-1};$$

accordingly,

$$u_n^m = (-)^m(n+\tfrac{1}{2})(n+\tfrac{3}{2})\ldots\ldots(n+m-\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots\ldots(n-m+\tfrac{1}{2})\,u_n.$$

The value of u_n is $\pi(\nu^2-1)^{-\frac{1}{2}}$ (see Hydrodynamics, §278, equations (63)).

The corresponding result for ordinary associated functions is

$$u_n^m = (-)^m (n+1)(n+2) \ldots . (n+m)\, n\, (n-1) \ldots . (n-m+1)\, u_n.$$

Here $u_n = (\nu^2 - 1)^{-1}$, so that

$$P_n'^m Q_n^m - P_n^m Q_n'^m = \frac{(-)^m (n+m)!}{(n-m)!\,(\nu^2-1)}.$$

10. Almost, if not all, the foregoing results are given in Prof. Hicks' paper, although some of them are exhibited in an erroneous form. But in physical investigations relating to the potentials of anchor rings, $\nu = \cosh \eta$, where the equation $\eta =$ const. represents a family of such surfaces, and in these investigations η is usually a very large quantity, and consequently $\varepsilon^{-\eta}$ is very small. Accordingly, if we put $\varepsilon^{-\eta} = k$, we can expand these functions in series of powers of k, and these series furnish expressions which are of great convenience in discussing the motion of circular vortices. We shall therefore proceed to investigate the appropriate series for the two kinds of zonal toroidal functions.

Putting $m = 0$ in (8) we obtain

$$Q_n = \int_0^\infty \frac{d\phi}{\{\nu + (\nu^2-1)^{\frac{1}{2}} \cosh \phi\}^{n+\frac{1}{2}}}.$$

In this write

$$\nu = \tfrac{1}{2}(k + k^{-1}),$$
$$pk = \{\nu + (\nu^2 - 1)^{\frac{1}{2}} \cosh \phi\}^{-1},$$

and the integral becomes

$$Q_n = k^{n+\frac{1}{2}} \int_0^1 \frac{p^{n-\frac{1}{2}}\,dp}{(1-p)^{\frac{1}{2}}(1-k^2p)^{\frac{1}{2}}}$$

$$= 2k^{n+\frac{1}{2}} \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n}\theta\, d\theta}{(1 - k^2 \sin^2\theta)^{\frac{1}{2}}}, \tag{22}$$

if $p = \sin^2\theta$.

From (22) it follows that

$$\left.\begin{aligned} Q_0 &= 2k^{\frac{1}{2}} F(k), \\ Q_1 &= 2k^{-\frac{1}{2}} (F - E), \end{aligned}\right\} \tag{23}$$

where F and E are the first and second elliptic integrals to modulus k.

whilst the three equations (19), (18) and (17) connecting the associated functions degree n and orders $m+1$, m and $m-1$, become

$$\left.\begin{aligned}(\nu^2-1)\frac{dP_n^m}{d\nu} &= (\nu^2-1)\,P_n^{m+1} + m\nu P_n^m,\\ (\nu^2-1)\frac{dP_n^m}{d\nu} &= (n+m)(n-m+1)(\nu^2-1)^{\frac{1}{2}}P_n^{m-1} - m\nu P_n^m,\\ P_n^{m+1} + \frac{2m\nu}{(\nu^2-1)^{\frac{1}{2}}}P_n^m &= (n+m)(n-m+1)\,P_n^{m-1}.\end{aligned}\right\} \qquad (21)$$

These formulæ can be proved directly from equation (5).

9. The definite integral expression (8) for toroidal functions of the second kind can be shown by a precisely similar method to satisfy the same equations, but in deducing equations such as (13) or (14) it must be recollected that m must be supposed to be not greater than $n-1$. Similar observations apply to the ordinary associated functions of the second kind, which can be deduced by writing $n+\frac{1}{2}$ for n.

There is one result which is of considerable importance in physical investigations, viz. the value of the quantity

$$P_n'^m Q_n^m - Q_n'^m P_n^m,$$

where the accents denote differentiation with respect to ν. Calling this u_n^m, substitute the values of $P_n'^m$ and $Q_n'^m$ from (18) and we get

$$u_n^m = \frac{(n+m-\frac{1}{2})(n-m+\frac{1}{2})}{(\nu^2-1)^{\frac{1}{2}}}\,(P_n^{m-1}Q_n^m - P_n^m Q_n^{m-1}).$$

In (19) write $m-1$ for m, and then substitute in the last equation the values of Q_n^m and P_n^m and we get

$$u_n^m = -(n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})\,u_n^{m-1};$$

accordingly,

$$u_n^m = (-)^m(n+\tfrac{1}{2})(n+\tfrac{3}{2})\ldots\ldots(n+m-\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots\ldots(n-m+\tfrac{1}{2})\,u_n.$$

The value of u_n is $\pi(\nu^2-1)^{-\frac{1}{2}}$ (see Hydrodynamics, §278, equations (63)).

The corresponding result for ordinary associated functions is

$$u_n^m = (-)^m (n+1)(n+2) \ldots . (n+m)\, n(n-1) \ldots . (n-m+1)\, u_n.$$

Here $u_n = (\nu^2 - 1)^{-1}$, so that

$$P_n'^m Q_n^m - P_n^m Q_n'^m = \frac{(-)^m (n+m)!}{(n-m)!\,(\nu^2-1)}.$$

10. Almost, if not all, the foregoing results are given in Prof. Hicks' paper, although some of them are exhibited in an erroneous form. But in physical investigations relating to the potentials of anchor rings, $\nu = \cosh\eta$, where the equation $\eta =$ const. represents a family of such surfaces, and in these investigations η is usually a very large quantity, and consequently $\epsilon^{-\eta}$ is very small. Accordingly, if we put $\epsilon^{-\eta} = k$, we can expand these functions in series of powers of k, and these series furnish expressions which are of great convenience in discussing the motion of circular vortices. We shall therefore proceed to investigate the appropriate series for the two kinds of zonal toroidal functions.

Putting $m = 0$ in (8) we obtain

$$Q_n = \int_0^\infty \frac{d\phi}{\{\nu + (\nu^2-1)^{\frac{1}{2}} \cosh\phi\}^{n+\frac{1}{2}}}.$$

In this write

$$\nu = \tfrac{1}{2}(k + k^{-1}),$$
$$pk = \{\nu + (\nu^2 - 1)^{\frac{1}{2}} \cosh\phi\}^{-1},$$

and the integral becomes

$$Q_n = k^{n+\frac{1}{2}} \int_0^1 \frac{p^{n-\frac{1}{2}}dp}{(1-p)^{\frac{1}{2}}(1-k^2p)^{\frac{1}{2}}}$$
$$= 2k^{n+\frac{1}{2}} \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n}\theta\, d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}}, \qquad (22)$$

if $p = \sin^2\theta$.

From (22) it follows that

$$\left.\begin{aligned} Q_0 &= 2k^{\frac{1}{2}}F(k), \\ Q_1 &= 2k^{-\frac{1}{2}}(F - E), \end{aligned}\right\} \qquad (23)$$

where F and E are the first and second elliptic integrals to modulus k.

Let H_s denote the coefficient of x^s in the expansion for $(1-x)^{-\frac{1}{2}}$, so that when s is not zero,

$$H_s = \frac{1.3.5 \ldots . (2s-1)}{2.4.6 \ldots . 2s},$$

whilst $H_0 = 1$; then since

$$\int_0^{\frac{1}{2}\pi} \sin^{2s}\theta d\theta = \tfrac{1}{2}\pi H_s,$$

we obtain
$$Q_n = \pi k^{n+\frac{1}{2}} \Sigma_{s=0}^{s=\infty} H_s H_{n+s} k^{2s}. \tag{24}$$

This is the series for Q_n in powers of k.

11. The function P_n is more difficult to deal with. From (7) we have

$$P_n = \int_0^\pi \frac{d\phi}{\{\nu + (\nu^2+1)^{\frac{1}{2}}\cos\phi\}^{n+\frac{1}{2}}},$$

and by means of a well-known transformation, this can be expressed in the form

$$P_n = \int_0^\pi \{\nu + (\nu^2 - 1)^{\frac{1}{2}}\cos\phi\}^{n-\frac{1}{2}} d\phi.$$

Putting $\nu = \frac{1}{2}(k + k^{-1})$, this becomes

$$P_n = 2k^{-n+\frac{1}{2}} \int_0^\pi (1 - k'^2 \sin^2\theta)^{n-\frac{1}{2}} d\theta, \tag{25}$$

from which it follows that

$$\left.\begin{aligned} P_0 &= 2k^{\frac{1}{2}} F(k'), \\ P_1 &= 2k^{-\frac{1}{2}} E(k'). \end{aligned}\right\} \tag{26}$$

From equation (25) it follows that P_n is a species of generalized elliptic integral; and since the elliptic integrals of the first and second kinds, when k' is nearly equal to unity, are known to be capable of expansion in a series of the form

$$u \log 4/k + v,$$

where u and v are series proceeding according to even powers of k^2, we should

anticipate that the definite integral in (25) is capable of being expressed in a similar manner.* This we shall now show to be the case.

In (9) put $m=0$, $\nu=\frac{1}{2}(k+k^{-1})$, and it becomes

$$k^2\frac{d^2u}{dk^2}-\frac{2k^3}{1-k^2}\frac{du}{dk}-(n^2-\tfrac{1}{4})u=0.$$

In this write $u=\eta k^{-n+\frac{1}{2}}$, and the equation for η is

$$(1-k^2)\frac{d^2\eta}{dk^2}-\{(2n-1)-(2n-3)k^2\}k^{-1}\frac{d\eta}{dk}+(2n-1)\eta=0. \qquad (27)$$

Assume $$\eta=\phi_n(k)\log 4/k+\psi_n(k). \qquad (28)$$

Substituting in (27), it will be found that the equation will be satisfied, provided ϕ_n satisfies an equation of the same form as (27), whilst ψ_n satisfies the equation

$$(1-k^2)\frac{d^2\psi}{dk^2}-\{2n-1-(2n-3)k^2\}k^{-1}\frac{d\psi}{dk}$$
$$+(2n-1)\psi-2(1-k^2)k^{-1}\frac{d\phi}{dk}+\{2n-(2n-2)k^2\}\frac{\phi}{k^2}=0. \qquad (29)$$

To find the value of ϕ_n, assume

$$\phi_n=\Sigma A_{2s}k^{2s},$$

and substitute in (27) and we get

$$2s(2s-2n)A_{2s}k^{2s-2}$$
$$-\{(2s+1)(2s-2n+1)A_{2s}-(2s+2)(2s-2n+2)A_{2s+2}\}k^{2s}-\ldots=0.$$

This equation will be satisfied provided the first term begins with $s=0$ or $s=n$; but inasmuch as it is known that in the expression for $E(k')$, the first term of

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* The neatest way that I have met with of proving the series for $F(k')$, when k' is nearly equal to unity, is contained in a paper by Prof. Sylvester, Phil. Mag., XX (1860), p. 528. His method leads to the remarkable result that

$$\int_0^{\frac{1}{2}\pi}\frac{\log\cos\phi\,d\phi}{(1-k^2\cos^2\phi)^{\frac{1}{2}}}=-\int_0^{\frac{1}{2}\pi}\frac{\log\{1+(1-k^2\cos^2\phi)^{\frac{1}{2}}\}d\phi}{(1-k^2\cos^2\phi)^{\frac{1}{2}}}.$$

the series by which $\log 4/k$ is multiplied is $\frac{1}{2}k^2$, it follows that we must take $s = n$, so that the first term in the series for ϕ_n is k^{2n}. Also

$$A_{2s+2} = \frac{(2s+1)(2s-2n+1)}{(2s+2)(2s-2n+2)} A_{2s},$$

whence

$$A_{2n+2s} = \frac{(2n+1)(2n+3) \ldots . (2n+2s-1)}{(2n+2)(2n+4) \ldots . (2n+2s)} H_s A_{2n}$$

and

$$\phi_n = A_{2n} k^{2n} \left\{ 1 + \frac{2n+1}{2.2n+2} k^2 + \frac{1.3.(2n+1)(2n+3)}{2.4.(2n+2)(2n+4)} k^4 + \ldots . \right\} \tag{30}$$

We shall now show that $A_{2n} = 2H_n$. Since ϕ_n satisfies the differential equation (27), it follows that $\phi_n k^{-n+\frac{1}{2}}$ satisfies the differential equation for zonal toroidal functions, and accordingly if A_{2n} be suitably chosen, the series (30) will satisfy the sequence equation (14). Putting $m = 0$ and $\nu = \frac{1}{2}(k + k^{-1})$, this equation becomes

$$(n + \tfrac{1}{2}) P_{n+1} - n(1 + k^2) k^{-1} P_n + (n - \tfrac{1}{2}) P_{n-1} = 0. \tag{31}$$

On substitution, this will be found to be satisfied provided $A_{2n} = 2H_n$; also if we put $n = 0$ and $n = 1$, it will be found that the series for ϕ_0 and ϕ_1 are respectively the coefficients of $\log 4/k$ in the series for $2F(k')$ and $2E(k')$. We therefore finally obtain for the portion of P_n which involves $\log 4/k$ the expression

$$2k^{n+\frac{1}{2}} \log 4/k \Sigma_{s=0}^{s=\infty} H_s H_{n+s} k^{2s} = (2/\pi) Q_n \log 4/k \tag{32}$$

by (24), whilst

$$\phi_n = 2k^{2n} \Sigma_{s=0}^{s=\infty} H_s H_{n+s} k^{2s}. \tag{33}$$

12. To obtain the series for ψ_n we must revert to the differential equation (29), and we must first arrange the last two terms in a series of powers of k. The series for ϕ_n may be written in the form

$$\phi_n = 2H_n k^{2n} \Sigma_1^\infty (1 + A_{2s} k^{2s}),$$

where

$$A_{2s} = H_s \frac{(2n+1)(2n+3) \ldots . (2n+2s-1)}{(2n+2)(2n+4) \ldots . (2n+2s)}. \tag{34}$$

Substituting in the last two terms of (29), and rearranging according to powers of k, the differential equation becomes

$$(1-k^2)\frac{d^2\psi}{dk^2} - \{2n-1-(2n-3)k^2\}\frac{1}{k}\frac{d\psi}{dk} + (2n-1)\psi$$
$$+ 2H_n\left[-2k^{2n-2} + \Sigma_0^\infty\left\{\frac{2n+2s+1}{2s+2} + \frac{2s+1}{2n+2s+2}\right\}A_{2s}k^{2n+2s}\right] = 0. \quad (35)$$

The case of $n=0$ would merely give us the series for $F(k')$ which is considered in all books on Elliptic Functions. When n is equal to unity, it is known from the series for $E(k')$ that the series for ψ begins with a term which is independent of k. We shall therefore assume that

$$\psi = \Sigma_0^\infty B_{2r}k^{2r}.$$

Substituting in (35) we get

$$\Sigma_0^\infty\{(2r+1)(2n-2r-1)B_{2r} - (2r+2)(2n-2r-2)B_{2r+2}\}k^{2r}$$
$$+ 2H_n\left[-2nk^{2n-2} + \Sigma_0^\infty\left\{\frac{2n+2s+1}{2s+2} + \frac{2s+1}{2n+2s+2}\right\}A_{2s}k^{2n+2s}\right] = 0. \quad (36)$$

If, therefore, r is not greater than $n-1$, which can only happen when n is not less than unity, we get

$$B_{2r+2} = \frac{(2r+1)(2n-2r-1)}{(2r+2)(2n-2r-2)}B_{2r}. \quad (37)$$

We therefore see that ψ consists of two parts, viz. a terminating series the law of whose coefficients is independent of ϕ, and whose values are given by (37), and an infinite series which we shall presently investigate. Putting $r=n-1$ in (36), we see that

$$B_{2n-2} = 2H_{n-1},$$

from which it follows that the terminating series is

$$\frac{2}{H_{n-1}}\left\{1 + \frac{2n-1}{2(2n-2)}k^2 + \frac{1.3.(2n-1)(2n-3)}{2.4.(2n-2)(2n-4)}k^4 + \ldots\right.$$
$$\left. + H_{n-2}\frac{(2n-1)(2n-3)\ldots.7.5}{(2n-2)(2n-4)\ldots.6.4}k^{2n-4} + H_{n-1}^2k^{2n-2}\right\}. \quad (38)$$

To find the infinite series, put $r = n$ in (36), and recollecting the value of A_{2s}, we get

$$B_{2n+2} = \frac{2n+1}{2.2n+2} B_{2n} - H_{n+1}\left\{\frac{1}{1.2} + \frac{1}{(2n+1)(2n+2)}\right\}, \tag{39}$$

and generally

$$B_{2n+2s+2} = \frac{(2s+1)(2n+2s+1)}{(2s+2)(2n+2s+2)} B_{2n+2s}$$
$$- 2H_{s+1}H_{n+s+1}\left\{\frac{1}{(2s+1)(2s+2)} + \frac{1}{(2n+2s+1)(2n+2s+2)}\right\}. \tag{40}$$

13. Equation (40) enables us to obtain all the coefficients in terms of B_{2n} which is undetermined, and we shall now proceed to find its value.

The portion of P_n which involves $\log 4/k$ is $\phi_n k^{-n+\frac{1}{2}} \log 4/k$; and since by (32) and (33) this is equal to $(2/\pi)\, Q_n \log 4/k$, it follows that $\phi_n k^{-n+\frac{1}{2}}$ satisfies the sequence equation (31), as can be readily verified by actual substitution; whence $\psi_n k^{-n+\frac{1}{2}}$ also satisfies (31), and consequently ψ_n satisfies the equation

$$(2n+1)\,\psi_{n+1} - 2n(1+k^2)\,\psi_n + (2n-1)\,k^2\psi_{n-1} = 0. \tag{41}$$

In this write $\psi_n = \Sigma_{r=0}^{r=\infty} B_{2n}^n k^{2r}$,

where the index n denotes the *degree* of the function to which B belongs, whilst the suffix $2r$ denotes the *power* of k of which it is the coefficient. The quantity which we require to determine is the coefficient of k^{2n} in the series for ψ_n, and in this notation it is represented by B_{2n}^n.

Substituting the above value of ψ_n in (41), and equating coefficients of k^{2r}, we get

$$(2n+1)\,B_{2r}^{n+1} - 2n\,(B_{2r}^n + B_{2r-2}^n) + (2n-1)\,B_{2r-2}^{n-1} = 0. \tag{42}$$

We have already shown that $B_{2n-2}^n = 2H_{n-1}$, whence, writing $r = n$ in (42), we get

$$B_{2n}^n = \frac{2n-1}{2n} B_{2n-2}^{n-1} - \frac{2H_n}{2n\,(2n-1)}. \tag{43}$$

Now $P_1 = 2k^{-\frac{1}{2}}E(k') = k^{-\frac{1}{2}}(\phi_1 \log 4/k + \psi_1)$,

and consequently B_2^1, which is the coefficient of k^2 in the expansion of ψ_1, is

equal to twice the coefficient of k^2 in that portion of the series for $E(k')$ which does not involve $\log 4/k$; accordingly

$$B_2^1 = -\tfrac{1}{2},$$

whence by (43) we obtain

$$\begin{aligned} B_{2n}^n &= -2H_n\left\{\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots . \frac{1}{(2n-1)\,2n}\right\} \qquad (44) \\ &= -2H_nS_n, \end{aligned}$$

where S_n denotes the above series. Going back to (40) and using this value of B_{2n}^n, we see that we may write

$$B_{2n+2s}^n = -2H_sH_{n+s}(S_s + S_{n+s}),$$

in which the symbols H_0 and S_0 must be interpreted as being respectively equal to unity and zero.

The second part of the series for ψ_n is therefore equal to

$$-2\Sigma_{s=0}^{s=\infty}H_sH_{n+s}(S_s + S_{n+s})\,k^{2n+2s},$$

and the value of P_n finally becomes

$$\begin{aligned} P_n = {} & 2k^{n+\frac{1}{2}}\log 4/k\Sigma_{s=0}^{s=\infty}H_sH_{n+s}k^{2s} \\ & + \frac{2}{H_{n-1}}\left\{1 + \frac{2n-1}{2.2n-2}k^2 + \frac{1.3.2n-1.2n-3}{2.4.2n-2.2n-4}k^4 + \ldots . \right. \\ & \left. + H_{n-2}\frac{2n-1.2n-3\ldots . 7.5}{2n-2.2n-4\ldots . 6.4}k^{2n-4} + H_{n-1}^2k^{2n-2}\right\}k^{-n+\frac{1}{2}} \\ & - 2k^{n+\frac{1}{2}}\Sigma_{s=0}^{s=\infty}H_sH_{n+s}(S_s + S_{n+s})\,k^{2s}. \qquad (45) \end{aligned}$$

This series holds good for all positive integral values of n, *excluding* zero. When $n=0$, the value of P_0 may be deduced from the known series for $F(k')$, and when $n=1$, the series (45) reduces to the known series for $2k^{-\frac{1}{2}}E(k')$.

The first few terms of the series for the first five zonal functions are given by Prof. Hicks,* but as he has obtained them by direct calculation from the sequence equation (31) combined with the series for $F(k')$ and $E(k')$, his results do not show the law of formation of the coefficients.

14. It is quite obvious that the definite integrals (7) and (8), which give the values of the functions of any order and degree, could be expressed by

* Phil. Trans., 1884, pp. 171, 172.

series analogous to (45) and (24), but the results would be of little physical interest, as the utility of toroidal functions consists in their applications to the motion of circular vortex rings. The motion in most cases of practical interest can be obtained by means of Stokes' current function; and this function, as is well known, can be made to depend on a solution of Laplace's equation which involves harmonics of degree n and order unity. We therefore require to investigate the functions P_n^1, Q_n^1. Putting $m = 0$ in (11) we obtain

$$(\nu^2 - 1)^{\frac{1}{2}} P_n^1 = (n + \tfrac{1}{2})(P_{n+1} - \nu P_n), \tag{46}$$

with a similar equation for Q. Now the functions which are most useful in investigating the motion of circular vortex rings are two functions R and T (which are different from Prof. Hicks' R and T), and satisfy the equations*

$$(\nu^2 - 1)^{\frac{1}{2}} P_n^1 = k^{-n-\frac{1}{2}} R_n,$$
$$(\nu^2 - 1)^{\frac{1}{2}} Q_n^1 = -\tfrac{1}{4}\pi k^{n-\frac{1}{2}} T_n,$$

so that from (46) we get

$$R_n = \tfrac{1}{4}(2n + 1)\{2kP_{n+1} - (1 + k^2) P_n\} k^{n-\frac{1}{2}},$$
$$T_n = -\pi^{-1}(2n + 1)\{2kQ_{n+1} - (1 + k^2) Q_n\} k^{-n-\frac{1}{2}}.$$

Since only a few terms of the series for R and T are required in physical investigations, they may be easily calculated from the above formulæ by means of (45) and (24).

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* See my Treatise on Hydrodynamics, Chap. XII, equations (65) and (71).

Simple Groups as far as Order 660.

BY F. N. COLE, PH. D.

1. In an earlier paper* I have shown that the orders of simple groups between the limits 200 and 500 are restricted to two possibilities, 360 and 432. In the following it is shown that the latter of these two orders is inadmissible, and that the former furnishes only one type of a simple group. On continuing the census as far as order 660, two other simple groups present themselves, of orders 504 and 660 respectively. Of these the second is familiar from the theory of the elliptic modular functions. That of order 504, however, seems hardly to have been recognized hitherto.

The list of simple groups of compound orders to order 660, as determined by Dr. Hölder and myself, consists of one type for each of the orders 60, 168, 360, 504, and 660.

I.—*First Reduction of Possible Orders.*

2. A preliminary survey of the numbers from 500 to 660 shows that the orders of simple groups between these limits are restricted to the following 13 possibilities:

$504 = 2^3.3^2.7,$	$540 = 2^2.3^3.5,$	$612 = 2^2.3^2.17,$
$520 = 2^3.5\,.13,$	$546 = 2\,.3\,.7.13,$	$616 = 2^3.7\,.11,$
$525 = 3\,.5^2.7,$	$552 = 2^3.3\,.23\,,$	$630 = 2\,.3^2.5\,.7,$
$528 = 2^4.3\,.11\,,$	$560 = 2^4.5\,.7,$	$660 = 2^2.3\,.5\,.11.$
	$576 = 2^6.3^2,$	

To these we add the orders left unconsidered in the former paper,

$$360 = 2^3.3^2.5, \quad 432 = 2^4.3^3.$$

* Amer. Jour., Vol. XIV, pp. 378–88.

3. The following four cases are immediately rejected:

520, 546, 552, 616,

inasmuch as the subgroups indicated by Sylow's theorem in each case would furnish a number of operations exceeding the order of the group. Thus a simple group of order 520 must contain 40 subgroups of order 13 and 26 of order 5; one of order 546 must contain 78 subgroups of order 7 and 14 of order 13; one of order 552 must contain 24 subgroups of order 23 and 46 of order 3; finally, one of order 616 must contain 56 subgroups of order 11 and 8 of order 7, leaving in this case only 8 operations remaining, from which only *one* subgroup of order 8 could be formed.

4. Of the 11 remaining cases the following 5 also present no difficulty:

525, 528, 540, 560, 576.

A simple group of order 525 must contain 15 subgroups of order 7. The isomorphic substitution group of 15 letters would contain 15 subgroups of order 35 affecting 14 letters each. But a group of order 35 is cyclical, and its generating substitutions must here consist of one cycle of 5 letters with one cycle of 7 letters. The group would therefore contain circular substitutions of 5 and of 7 letters, which are here impossible.

A simple group of order 528 must contain 12 subgroups of order 11. The isomorphic substitution group of 12 letters would contain 12 subgroups of order 44 affecting 11 letters each. The latter contain each a self-conjugate subgroup of order 11, and being therefore transitive, each contains subgroups of order 4 affecting 10 letters. The substitutions of the latter must be even, and none of them affect less than 5 letters. Then it is readily seen that they all affect 8 letters. But no such substitution can transform the circular substitutions of order 11 into any of their powers, as must here be the case.

A simple group of order 540 must contain 36 subgroups of order 5. The isomorphic substitution group of 36 letters would contain 36 subgroups of order 15 affecting 35 letters each. A group of order 15 is cyclical. Its substitutions of order 5 must here affect all the 35 letters, otherwise there would not be 36 *distinct* subgroups of order 5. Consequently the substitutions of order 15 must here consist of cycles of 15 letters with cycles of 5 letters, and we may have either 1 cycle of 15 letters with 4 of 5 letters or 2 cycles of 15 letters with 1

of 5 letters. But the group of order 15 contains only one subgroup of order 3, which will here affect 15 letters or 30 letters. The number of its conjugates would therefore be $\frac{36}{21}$ or $\frac{36}{6}$, both of which are impossible.

A simple group of order 560 must contain 8 subgroups of order 7. But a substitution group of order 560 and degree 8 would contain circular substitutions of order 5, and would accordingly be non-primitive or four-fold transitive, both of which possibilities are here excluded.

A simple group of order 576 would contain 9 subgroups of order 64 which, in the isomorphic substitution group of 9 letters, would affect 8 letters each. The substitutions of these subgroups must all be of order 2 or 4, for a circular substitution of 8 letters is odd. Rejecting all odd substitutions and those which affect less than 5 letters, we have left only such as affect each 8 letters. The 9 subgroups of order 64 then furnish $9.63 = 567$ distinct substitutions, leaving only 9, which can furnish only *one* subgroup of order 9.

5. Fairly simple considerations also suffice to dispose of the orders

$$432, \quad 612, \quad 630.$$

A simple group of order 432 must contain 16 subgroups of order 27 which, in the isomorphic substitution group of 16 letters, would affect 15 letters each. These subgroups of order 27 again contain subgroups of order 9, which are always commutative with the 27 operations of the group in which they occur, and which are therefore commutative within the entire group with a multiple of 27 operations. The number of conjugates within the entire group of such a subgroup of order 9 is therefore a divisor of $\frac{432}{27} = 16$. The only possibility here is 16 itself, since $8!$ is not divisible by 432. Consequently every subgroup of order 9 is commutative with exactly 27 operations, and therefore occurs in only one group of order 27. Then any two groups of order 27 can have only 1 or 3 operations in common, or, in other words, all but 1 or all but 3 of the substitutions of a group of order 27 affect all the 15 letters. In the former case the 16 subgroups of order 27 furnish $16.26 = 416$ distinct operations, leaving only 16, which can give only *one* subgroup of order 16. In the latter case the 3 substitutions of any group of order 27 which affect less than 15 letters form a

cyclical group affecting 3, 6, 9 or 12 letters, and having therefore $\frac{16}{13}$, $\frac{16}{10}$, $\frac{16}{7}$, or $\frac{16}{4}$ conjugates in the entire group, all of which suppositions are here impossible.

A simple group of order 612 must contain 18 subgroups of order 17. The isomorphic substitution group of 18 letters would contain 18 subgroups of order 34 affecting 17 letters each. The latter groups must here be non-cyclical; each of them contains 1 subgroup of order 17 and 17 subgroups of order 2, the latter affecting 16 letters each. There are no other substitutions of order 2 or of order 4 in the group. For if such were present they must affect 18 letters each. If of order 2 they would then be odd, and if of order 4, their squares must affect 16 letters, and they must consist therefore of 4 cycles of 4 letters with 1 cycle of two letters, being again odd. The operations of order 2 contained in the entire group are therefore all conjugate, and their number is $\frac{18}{2} \cdot 17 = 153$. The corresponding group of 612 substitutions of 153 letters would contain 153 subgroups of order 4 affecting 152 letters each. The actual substitutions of the latter groups are taken entirely from the 153 conjugate substitutions of order 2. These must all leave the same number of letters unchanged. If this number is x, the total number of substitutions of order 2 would be $\frac{153}{x} \cdot 3$. Hence $x = 3$, and every substitution of order 2 would affect 150 letters, and would therefore be odd.

A simple group of order 630 must contain either 21 or 126 subgroups of order 5. In the latter case the isomorphic substitution group of 126 letters would contain 126 subgroups of order 5 affecting 125 letters each, while all the other substitutions of the group would affect all the 126 letters. Among these there must be substitutions of order 2, which would then be odd. Accordingly a simple group of order 630 must contain exactly 21 subgroups of order 5. The isomorphic substitution group of 21 letters would then contain 21 subgroups of order 30 affecting 20 letters each. A group of order 30 contains a (cyclical) subgroup of order 15, the generating substitutions of which must here consist of 1 cycle of 15 letters and 1 of 5 letters. But a group of order 30 contains only one subgroup of order 3, which in the present case would affect 15 letters and would therefore have in the entire group $\frac{21}{6}$ conjugates.

II.—*The Simple Group of Order* 360.

6. A simple group of order 360 must contain either 6 or 36 subgroups of order 5. The former number is at once excluded, since the alternating substitution group of 6 letters contains 36 subgroups of order 5. Also the number of subgroups of order 9 is either 10 or 40. It will appear that only the former case actually occurs.

If 40 subgroups of order 9 were present, these could not be cyclical, for then they would furnish 40.6 distinct operations of order 9 in addition to the 36.4 operations of order 5 which are certainly present.

The 40 subgroups of order 9 must therefore contain each 4 subgroups of order 3. These 160 subgroups could not all be different, for then they would furnish 320 distinct operations. On being transformed by any one of the operations of order 3, the 40 subgroups of order 9 would be permuted among themselves, and $3x+1$ of them would be left unchanged. All those which contain the transforming operation of order 3 will certainly be left unchanged. And this is not the case with any of the others, for then the group of order 360 would contain subgroups of order 27. Accordingly some subgroup of order 3 is contained in $4, 7, 10, \ldots$ of the subgroups of order 9, and as it is commutative with all the operations of the latter, it is commutative with a subgroup whose order is a multiple of 4.9, 7.9, $10.9, \ldots$. Then the number of its conjugates in the entire group is a divisor of $\frac{360}{4.9}, \frac{360}{7.9}, \ldots$. The only admissible case is the first one. A subgroup of order 3 which occurs in more than one subgroup of order 9 occurs in exactly 4 of the latter, and has exactly 10 conjugates. It follows at once that

A simple group of order 360 *can always be expressed as a transitive substitution group of* 10 *letters.*

This group of 10 letters contains 10 conjugate subgroups of order 36 affecting 9 letters each. These contain each 1 or 4 subgroups of order 9. If these are transitive, the groups of order 36 contain each 9 subgroups of order 4 affecting 8 letters each. The substitutions of the latter being even, it is readily seen that every one of them, except identity, affects 8 letters. The group of order 36 would then contain 27 substitutions of order 2 and 4 and consequently only 1 subgroup of order 9.

On the other hand, of intransitive groups of order 9 affecting 9 letters or less and not containing any circular substitution of 3 letters, there is only one type, its 4 subgroups of order 3 being, for example,

$$\{(abc)(def)\},\ \{(abc)(ghi)\},\ \{(def)(gih)\},\ \{(abc)(dfe)(gih)\}.$$

Now, if the group of order 360 contained 40 subgroups of order 9, each subgroup of order 36 would contain 4 of these, and they must have, as shown above, one common subgroup of order 3. But it is immediately found on trial that it is impossible to construct more than 2 such groups of the preceding type. Also, if there were only 10 subgroups of order 9 in all, and if these were of the preceding type, the total number of subgroups of order 3 affecting 6 letters would be $\frac{10}{4}.3$. We have then the following result:

A simple group of order 360 *contains only* 10 *subgroups of order* 9, *and can be expressed as a doubly transitive substitution group of* 10 *letters.*

The transitive subgroups of order 36 which affect 9 letters contain, as just shown, 1 subgroup of order 9, all of whose substitutions except identity affect all the 9 letters, and 9 subgroups of order 4, all of whose substitutions except identity affect 8 letters. The subgroup of order 9 cannot be cyclical, for the 3 substitutions affecting a particular set of 8 letters cannot all transform a cycle of 9 letters into its powers, as is readily seen. Consequently the subgroup of order 9 contains 4 subgroups of order 3; for example,

$$\{(abc)(def)(ghi)\},\ \{(adg)(beh)(cfi)\},\ \{(aei)(bfg)(cdh)\},\ \{(afh)(bdi)(ceg)\}.$$

It will be clear, from an inspection of this group, that no substitution with 4 cycles of 2 letters each can transform any one of the subgroups of order 3 into another, and the only substitution of order 2 which leaves i, for example, unchanged and transforms each subgroup of order 3 into itself is

$$(ae)(bd)(cf)(gh).$$

Accordingly, the subgroups of order 4 which leave 2 letters unchanged are *cyclical*, and with the same notation as above, the group which leaves i and j unchanged may be assumed to be

$$\{(abed)(chfg)\}.$$

A simple group of order 360, as expressed in 10 letters, contains then 135 substitutions affecting 8 letters and 80 affecting 9 letters. These with the 144

substitutions of order 5 and identity make up the entire group. It only remains to be shown that the group can be constructed in essentially only one way.

There are 36 subgroups of order 5, and each of these is therefore transformed into itself by 10 substitutions. Of these, 5 must be selected from the substitutions of order 2. Moreover, the latter being all conjugate, every one of them transforms some subgroup of order 5 into itself, and therefore transforms some substitution of order 5 into its 4^{th} power. For the particular substitution of order 2,

$$(ae)(bd)(cf)(gh),$$

the corresponding substitution of order 5 must contain i and j in separate cycles. Moreover, if any of the letters a, b, e, d occur in the same cycle with i, a transformation by a power of $(abed)(chfg)$ will give a substitution of order 5 containing a and i in the same cycle and still transformed into its 4^{th} power by $(ae)(bd)(cf)(gh)$. In a proper power of the substitution of order 5, a will follow i. We have, therefore, as typical forms for the cycle containing i only,

$$(iabde),\quad (iahge),\quad (iacfe),\quad (icghf),$$
$$(iadbe),\quad (iaghe),\quad (iafce),\quad (ichgf).$$

The first two and the last two cases are to be rejected. For these the cycle containing i is transformed into one of its powers by the substitution $(abed)(chfg)$. If the second cycle in each case is transformed at the same time into the same power, then the group of order 5 which the substitution of order 5 generates is transformed into itself by 20 substitutions, which is not permissible. And otherwise a proper combination of the substitution of order 5 with its transformed substitution would affect only 5 letters, which is also impossible.

Again, if we multiply the 4 remaining types at the left by $(aie)(bgf)(chd)$, we obtain substitutions which do not affect a or i, and which therefore consist of 4 cycles of 2 letters or 2 cycles of 4 letters. For portions of these substitutions we have

$$(beh\,..)(cg\,....),\quad (gec\,..)(df\,....),$$
$$(ceg\,..)(bh\,....),\quad (def\,..)(gc\,....),$$

and considering the various forms of the second cycle of 5 letters, we find only the following possibilities for the typical substitutions of order 5:

$$(iahge)(jbcfd),\quad (iacfe)(jgdbh),$$
$$(iaghe)(jcbdf),\quad (iafce)(jdghb).$$

Of these, the first and the fourth cases lead to rejected types; for we have

$$(acb)(dfe)(gih).(iahge)(jbcfd) = (afig)(bhej),$$
$$[(afig)(bhej)]^2.(ae)(bd)(cf)(gh) = (iedba)(jgefh),$$

and

$$(agd)(bhe)(cif).(iafce)(jdghb) = (ahic)(dfej),$$
$$[(ahic)(dfej)]^2.(ae)(bd)(cf)(gh) = (iebda)(jcghf).$$

The two remaining cases are identical. For

$$(abc)(def)(ghi).(iaghe)(jcbdf) = ihad.jcge,$$
$$[(ihad)(jcge)]^2.(ae)(bd)(cf)(gh) = (iefca)(jhbdg).$$

There is then only one simple group of order 360. *This is therefore isomorphic with the alternating substitution group of* 6 *letters.*

III.—*The Simple Group of Order* 504.

A simple group of order 504 must contain either 8 or 36 subgroups of order 7. In the former case the isomorphic substitution group of 8 letters would contain subgroups of order 63 affecting 7 letters. The latter would contain subgroups of order 9, which would here include circular substitutions of 3 letters. There must, therefore, be 36 subgroups of order 7.

The isomorphic substitution group of 36 letters contains 36 subgroups of order 14 affecting 35 letters each. These contain subgroups of order 7 which must affect all the 35 letters, since otherwise there would not be 36 distinct subgroups of order 7. If the groups of order 14 were cyclical, their generating substitutions, being even, must consist of 2 cycles of 14 letters and 1 cycle of 7 letters. Such a group would contain one substitution of order 2 affecting 28 letters, and having therefore in the entire group $\frac{36}{8}$ conjugates. Accordingly, the groups of order 14 are non-cyclical, and each of them contains, beside the subgroup of order 7, 7 conjugate substitutions of order 2, which transform the substitutions of order 7 into their 6^{th} powers. The substitutions of order 2 might leave one letter of each of the 5 cycles of a substitution of order 7 unchanged and permute the 4 remaining letters of each cycle in pairs, or they might interchange the letters of one or of two pairs of cycles. It appears that an *even* substitution of order 2 must leave exactly 4 letters unchanged. The total number of conjugate substitutions of this type is therefore $\frac{36}{4}.7 = 63$.

Of subgroups of order 9 there must be either 7 or 28, and it appears at once that only the latter number is admissible. The isomorphic substitution group of 28 letters contains 28 subgroups of order 18 affecting 27 letters. Each of these subgroups of order 18 contains a single subgroup of order 9. If all the substitutions of the latter, except identity, affect all the 27 letters, these groups would furnish $28.8 = 224$ substitutions of order 3 or 9, and these with the $36.6 = 216$ of order 7, the 63 of order 2, and identity would make up the entire group.

If any of the substitutions of a group of order 9 affect less than 27 letters, the group cannot be cyclical. For then the subgroup affecting less than 27 letters could only be of order 3 and affect 9 or 18 letters, and such a subgroup would have $\frac{28}{19}$ or $\frac{28}{10}$ conjugates in the entire group. Again, if the groups of order 9 were non-cyclical, a subgroup which affected less than 27 letters could affect only 24, 21, 18, ... letters. It would then occur in 4, 7, 10, ... of the groups of order 9, and would therefore be commutative with a multiple of 4.9, 7.9, 10.9, ... substitutions of the entire group. Then its number of conjugates in the latter would be a divisor of $\frac{504}{4.9}$, $\frac{504}{7.9}$, ... The only admissible possibility is that it should occur in 4 groups of order 9 and have 14 conjugates.

The isomorphic substitution group of 14 letters would contain 14 subgroups of order 36, each including 4 non-cyclical subgroups of order 3. The latter have a subgroup of order 3 in common, and furnish beside this $4.3 = 12$ other subgroups of order 3. Among the remaining substitutions of the groups of order 36 must be contained the 63 conjugate substitutions of order 2. These, being even, must affect either 8 or 12 letters. If there are h of them in each group, the number of their conjugates is $63 = \frac{14h}{6}, \frac{14h}{2}$; hence $h = 27, 9$. Only the latter case is possible, and the groups of order 36 are then complete.

From the preceding considerations we deduce the result that

In a simple group of order 504 *the operations of order* 2 *are all conjugate, and their number is* 63.

The isomorphic substitution group of 63 letters contains 63 subgroups of order 8 affecting 62 letters each. These groups contain the 63 substitutions of order 2, the existence of which has just been demonstrated. If h of these occur

in each group of order 8, and if each of them leaves k letters unchanged, their total number is $63 = \frac{63h}{k}$. Consequently $h = k$. But as $h < 8$ and the substitutions of order 2 are even, we can only take $k = 3, 7$.

In the latter case the groups of order 8 are made up entirely from the substitutions of order 2. These are all commutative with each other, and as each of them occurs in 7 of the groups of order 8, each is commutative with more than 8 substitutions, and has therefore less than 63 conjugates, *unless the* 7 *groups in which it occurs all coincide, so that there are only* 9 *distinct groups of order* 8.

Again, if $k = 3$, each group of order 8 contains 4 as yet unidentified substitutions. These could only be of order 4, and as their squares, being of order 2, must affect 60 letters, they must consist of 15 cycles of 4 letters and 1 cycle of 2 letters. But then they affect 62 letters, and the entire group would contain $63.4 = 252$ of them, which is impossible.

Accordingly a simple group of order 504 *contains exactly* 9 *subgroups of order* 8.

Expressed as a substitution group of 9 letters, the group contains circular substitutions of 7 letters, and is therefore triply transitive. The 9 doubly transitive subgroups which affect 8 letters each are of order 56, and each of them contains 8 circular subgroups of order 7 and a single subgroup of order 8. The latter, as already shown, contains, beside identity, 7 substitutions of order 2 which here affect 8 letters each. The substitutions of order 7 transform the 7 substitutions of order 2 in a cycle.

For the group of order 8 we may take

$$\begin{aligned}1,\ & (ab)(cd)(ef)(gh),\\ & (ac)(bd)(eg)(fh),\\ & (ad)(bc)(eh)(fg),\\ & (ae)(bf)(cg)(dh),\\ & (af)(be)(ch)(dg),\\ & (ag)(bh)(ce)(df),\\ & (ah)(bg)(cf)(de),\end{aligned}$$

and for one of the substitutions of order 7,

$$\sigma = (bcedghf).$$

The substitution σ transforms the group of order 8 into itself, and in combination

with the latter therefore generates one of the required groups of order 56 affecting 8 letters. As the latter can obviously be constructed in essentially only one way, it only remains to show that the substitutions of 9 letters can also be chosen in only one way, and that the result is actually a group.

For this purpose it is convenient to recur to the fact that the group of order 504 contains 36 subgroups of order 7, each of which is therefore transformed into itself by 14 substitutions. These form a non-cyclical group of this order which therefore contains 7 substitutions of order 2, which transform the corresponding substitution of order 7 into its 6^{th} power. For the particular substitution

$$\sigma = (bcedghf),$$

one of the required substitutions of order 2 is

$$t = (cf)(eh)(dg)(ai),$$

and the combination of this with $u = (ac)(bd)(eg)(fh)$ gives

$$tu = \rho = (aichgbdef).$$

The last substitution combined with the group of order 56 above gives rise to a group the order of which is a multiple of $9.56 = 504$.

The group of order 56 being denoted by H, it can now be shown that for every α

$$\rho^\alpha H = H\rho^\beta.$$

The order of the group generated by ρ and H is then exactly 504. Writing

$$\begin{array}{lll} \tau_2 = (ab)(cd)(ef)(gh), & \tau_5 = (ae)(bf)(cg)(dh), & \tau_8 = (ah)(bg)(cf)(de), \\ \tau_3 = (ac)(bd)(eg)(fh), & \tau_6 = (af)(be)(ch)(dg), & \sigma = (bcedghf), \\ \tau_4 = (ad)(bc)(eh)(fg), & \tau_7 = (ag)(bh)(ce)(df), & \rho = (aichgbdef), \end{array}$$

and noticing that σ and any τ, as τ_3, suffice to generate H, we have only to show that for every α

$$\rho^\alpha\tau_3 = H\rho^\beta, \quad \rho^\alpha\sigma = H\rho^\gamma.$$

But we have at once

$$\tau_3^{-1}\rho\tau_3 = \rho^{-1}, \quad \rho^\alpha\tau_3 = \tau_3\rho^{-\alpha},$$

and a brief calculation furnishes

$$\begin{array}{ll} \rho\sigma = \sigma\tau_7\rho^6, & \rho^5\sigma = \tau_6\rho^3, \\ \rho^2\sigma = \sigma^6\rho^7, & \rho^6\sigma = \sigma^5\tau_6\rho^5, \\ \rho^3\sigma = \sigma^2\tau_4\rho^2, & \rho^7\sigma = \sigma^2\tau_3\rho^4, \\ \rho^4\sigma = \sigma^5\tau_3\rho, & \rho^8\sigma = \sigma^6\tau_4\rho^8. \end{array}$$

The existence of the group of order 504 having now been demonstrated, it is easy to show that this group is in reality simple. Those of its substitutions which affect less than 9 letters are already known. They are 216 substitutions of order 7 and 63 of order 2, the latter affecting 8 letters each. The remaining 224 substitutions affecting 9 letters must all be regular, and are therefore all of order 3 or order 9. They are all certainly contained in subgroups of order 9. But the number of these subgroups cannot exceed 28, and in this case they can furnish 224 substitutions only if they have no substitutions common to two of them. There are, therefore, in fact 28 subgroups of order 9. As one of these, $\{\rho\}$, is cyclical, they are all cyclical, and, as just shown, their substitutions are all different.

If, now, the group contained a self-conjugate subgroup, the latter must be transitive. Then its order must be a multiple of 9, and accordingly it would contain a subgroup of order 9 and therefore all the subgroups of order 9. Its order, being a divisor of 504, and being as great as 224, must therefore be 252. But the 27 remaining substitutions could not be taken either from those of order 2 or those of order 7; for if any one of either set occurs in a self-conjugate subgroup, all of the set must occur. There is, therefore, no self-conjugate subgroup. It appears then that

There is one, and only one, simple group of order 504. *This group can be expressed as a transitive substitution group of* 9 *letters, and in this form it can be generated by the substitutions*

$$(ab)(cd)(ef)(gh), \quad (bcedghf), \quad (aichgbdef).$$

It contains 28 *cyclical subgroups of order* 9, *having no operation except identity common to any two of them;* 36 *subgroups of order* 7; *and* 9 *subgroups of order* 8, *each consisting of* 7 *substitutions of order* 2 *and identity.*

IV.—*The Simple Group of Order* 660.

This group must contain exactly 12 subgroups of order 11. The isomorphic substitution group of 12 letters contains 12 subgroups of order 55 affecting 11 letters each. Each of these subgroups contains 11 conjugate subgroups of order 5 affecting 10 letters each. The total number of subgroups of order 5 in the entire group is therefore $\frac{12}{2}.11 = 66$, and each of these is transformed into itself by a group of 10 substitutions.

These last groups cannot be cyclical, for there would be 66 of them, and they would furnish 66.4 new operations of order 10. There would then remain only 9 unidentified operations, and these could include at the most only 4 subgroups of order 3. Accordingly, each of the subgroups of order 10 contains 5 substitutions of order 2, which must here affect 12 letters each. These substitutions transform those of order 5 contained in the same group of order 10 into their 4^{th} powers. If, now, the substitution of order 5 is, for example,

$$s = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10),$$

the corresponding substitutions of order 2, which must affect 12 letters and transform s into its 4^{th} power, can only be

$$\begin{gathered}(1\ 6)\ (2\ 10)\ (3\ 9)\ (4\ 8)\ (5\ 7)\ (11\ 12),\\ (1\ 7)\ (2\ 6)\ (3\ 10)\ (4\ 9)\ (5\ 8)\ (11\ 12),\\ (1\ 8)\ (2\ 7)\ (3\ 6)\ (4\ 10)\ (5\ 9)\ (11\ 12),\\ (1\ 9)\ (2\ 8)\ (3\ 7)\ (4\ 6)\ (5\ 10)\ (11\ 12),\\ (1\ 10)\ (2\ 9)\ (3\ 8)\ (4\ 7)\ (5\ 6)\ (11\ 12).\end{gathered}$$

As soon, therefore, as a circular substitution of order 11 is assigned, one of order 5 is fixed and with it 5 of order 2. These are sufficient to generate the entire group, and there can therefore be only one type of a simple group of this order. That such a type exists is well known (cf. for example Klein, Elliptische Modulfunctionen, I, Chap. VIII).

ANN ARBOR, *May*, 1893.

On the Expansion of Functions in Infinite Series.

By W. H. Echols.

Cauchy's methods leave little to be desired in the expansion of functions in infinite series, and have transferred all at once the treatment of the subject from the methods of the Differential to those of the Integral Calculus. This is justly so, on account of the increased rigor of the analysis applied and on account of the facilities which it offers for the study of the properties of functions. Yet it has seemed to me not uninteresting to seek generalization of the older methods and to question wherein they have failed to supply the demand made of them. The following is a brief outline sketch made along the lines of enquiry, and imperfect as it may appear here and there in points but lightly touched, it seems to throw a light upon this subject from a different source, and it would appear that there may be certain points of vantage in the serial coefficients over their integral forms in discussing certain properties of functions.

1. Let fz and $\phi_r z\,(r = 1 \ldots . n - 1)$ be monogenic functions of a complex variable z, which are holomorphous throughout a certain region containing the arbitrary closed path C.

Let there be a function

$$S_n z \equiv \sum_{r=0}^{n-1} A_r \phi_r z,$$

in which $\phi_0 z \equiv 1$ and $\phi_r z$ are functions whose law of successive formation with respect to r is known, and whose coefficients A_r, independent of the variable z, are at present arbitrary, but whose law of formation is to be determined.

Let $R_1, \ldots, R_n$ be the differences between the functions fz and $S_n z$ at the points $z_1, \ldots, z_n$. We then have in

$$fz_m = S_n z_m + R_m, \qquad (m = 1 \ldots . n)$$

n relations connecting the n values A_r $(r = 0 \ldots . n - 1)$. Let us so determine these n quantities A_r that we shall have $R_m = 0$. This condition, which assures the arithmetical coincidence of the functions fz and $S_n z$ at the points $z_{1 \ldots . n}$, is expressed by*

$$A_r = \frac{|\phi_0 z_1 \ldots . \phi_{r-1} z_r . f z_{r+1} . \phi_{r+1} z_{r+2} \ldots . \phi_{n-1} z_n|}{|\phi_0 z_1 \ldots . \phi_{n-1} z_n|}. \tag{1}$$

Let the n points $z_{1 \ldots . n}$ be on the path C so that these points are at the vertices of an inscribed polygon of n sides $w_1 \ldots . w_n$. We then have generally

$$z_{r+1} - z_r = w_r \quad \text{and} \quad z_1 + \sum_1^n w_r = z_1.$$

We suppose w_r $(r = 1 \ldots . n)$ to be such that mod w_r vanishes along with $1/n$. The condition that fz and $S_n z$ shall coincide at an infinite number of consecutive points along C, that is, all along the path C, is that the coefficient A_r shall have the limiting value of this ratio (1) when $n = \infty$. This ratio takes the indeterminate form $0/0$ when $n = \infty$, since then w_r is infinitesimal and z_p and z_{n-p} converge to z_1. To remove this indetermination we apply to each term of the ratio the operator

$$\left(\frac{d}{dz_2}\right)^1_{z_2 = z_1} \ldots . \left(\frac{d}{dz_n}\right)^{n-1}_{z_n = z_1}. \tag{2}$$

Whence

$$A_r = \mathop{\pounds}_{n=\infty} \frac{|\phi_1' z_1 \ldots . \phi_{r-1}^{r-1} z_1 . f^r z_1 . \phi_{r+1}^{r+1} z_1 \ldots . \phi_{n-1}^{n-1} z_1|}{|\phi_1' z_1 \ldots . \phi_{n-1}^{n-1} z_1|}, \tag{3}$$

and its limit may be taken without indetermination.

Let z be an arbitrary point taken anywhere on C at a finite distance from z_1. The condition (3) is independent of the route pursued in tracing the closed path C through z_1 and z, since the intermediate points no longer appear and the derivatives of the monogenic functions at z_1 are independent of the direction with which C passes through z_1, and since the functions are supposed uniform, their values at z_1 and z are also independent of the path traced between z_1 and z. This being so, condition (3) expresses, therefore, also the more general condition that the functions fz and $S_\infty z$ shall coincide at all points throughout

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*** The vertical space rule being used here to denote the umbral determinant notation and not the modulus of the complex quantity. The terms between the verticals being those of the principal diagonal.**

the whole region of holomorphism. If, therefore, the condition (3) be found to be such that $S_\infty z$ is a convergent series, then the arithmetical equivalence of this series with the function fz at all points throughout the holomorphic region is assured.

For example, if $\phi_r z \equiv z^r/r!$, the value of $S_\infty z$ is readily seen to be

$$\sum_0^\infty (z - z_1)^r f^r z_1 / r!,$$

the convergence of which is assured when the derivatives of finite order of fz at z_1 are not infinite, and after some fixed m^{th} derivative the $(m+p)^{\text{th}}$ derivative is not greater than the $(m+p)^{\text{th}}$ power of a finite quantity. If we may identify such properties to be those of a holomorphic function, then every holomorphic function is expansible in Taylor's series throughout its region of holomorphism. Let us regard the functions with which we are going to deal as enjoying these properties, then all properties which may be deduced from Taylor's series belong to them.

2. If we stop this process at the n^{th} term we obtain an interpolation formula for fz along the chosen path C expressed in terms of the functions at the points $z_{1 \ldots n}$ along C.

The difference R between the function fz and the interpolation value $S_n z$ at any arbitrary running point z on C will be

$$\frac{|fz,\ \phi_0 z_1 \ldots \phi_{n-1} z_n|}{|\phi_0 z_1 \ldots \phi_{n-1} z_n|} = R. \tag{4}$$

We may, if we choose, regard the points $z_{1 \ldots n}$ as being arbitrarily located isolated points about in the holomorphic region, and z as before a wandering point in that region. If we wish to evaluate this ratio when $n = \infty$, we may remove the indetermination caused by the convergence of the z's to z_1 by applying the operator (2) to each term. Regarding the z's as shrunk upon z_1, we have without indetermination the interpolation

$$\frac{\begin{vmatrix} fz & 1 & \phi_1 z & \ldots \phi_{n-1} z \\ fz_1 & 1 & \phi_1 z_1 & \ldots \phi_{n-} z_1 \\ f' z_1 & 0 & \phi_1' z_1 & \ldots \phi_{n-1}' z_1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{n-1} z_1 & 0 & \phi_1^{n-1} z_1 & \ldots \phi_{n-1}^{n-1} z_1 \end{vmatrix}}{|\phi_1' z_1 \ldots \phi_{n-1}^{n-1} z_1|} = R, \tag{5}$$

or

$$fz = \sum_{0}^{n-1} A_r \phi_r z + R.$$

*3. Let ψz be a function having the same properties with fz, and let us seek to compare their interpolative differences as obtained from (5), which we call R_ψ and R_f.

If Fz be a holomorphous function throughout a connected region containing the two points z' and z'', and if these points be zeros of Fz, then the derivative of Fz will have a zero somewhere on a finite path traced in this region between z' and z'', by the theorem of mean value

$$Fz' - Fz'' = \lambda S F'\zeta,$$

mod $\lambda \leqq 1$, $S =$ length of path, and ζ a point on this path between z' and z''.

Let Z be the ratio of R_f to R_ψ, which is to be determined. Let z_0 be some arbitrarily chosen fixed point on C. Then we have $Z_0 = R_{f_0}/R_{\psi_0}$. Consider the function

$$R_f - Z_0 R_\psi .$$

This and its first $n-1$ derivatives have the zero z_1. The function also has the zero z_0, therefore if the theorem of mean value holds good for it and its first $n-1$ derivatives, its n^{th} derivative must have a zero ζ somewhere on C. Whence

$$D^n_{z=\zeta} R_f - Z_0 D^n_{z=\zeta} R_\psi = 0,$$

and if the n^{th} derivative of R_ψ does not vanish in the region considered we may divide by it and have

$$R_f / R_\psi = D^n_{z=\zeta} R_f / D^n_{z=\zeta} R_\psi ,$$

since we may drop the subscript, z_0 being an arbitrary point. If, therefore, we be examining the expansion of an unfamiliar function fz, we may investigate it through the aid of a more familiar one, ψz. The remainder after the n^{th} term of the interpolation being

$$R_f = R_\psi \frac{D^n_{z=\zeta} R_f}{D^n_{z=\zeta} R_\psi} . \tag{6}$$

If ψz be expansible to infinity, then R_ψ vanishes with $1/n$ and we are concerned with the limit, when $n = \infty$, of the ratio of the n^{th} derivatives of

$$fz - \sum_{0}^{n-1} A_r \phi_r z \quad \text{and} \quad \psi z - \sum_{0}^{n-1} A_r \phi_r z$$

in the region under consideration.

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§3 applies only to functions of a real variable. $\lambda \equiv 1$.

If we take $\psi z \equiv \phi_n z$, then in the example given above of Taylor's series we have

$$R_f = \frac{(z - z_1)^n}{n!} f^n \zeta,$$
$$= \lambda^n S^n f^n \zeta / n!$$

Any number of apparently more general forms of this remainder can be made from the general theorem (6). Thus put R_f^n and R_ψ^p for the remainders after n and p terms of expansion of fz and ψz. Then

$$R_f = R_\psi^p \frac{D^p_{z=\zeta'} R^p_{\phi_p}}{D^p_{z=\zeta} R^n_{\phi_n}} \frac{D^n_{z=\zeta} R^n_f}{D^n_{z=\zeta'} R^p_\psi} \frac{R^n_{\phi_n}}{R^p_{\phi_p}}.$$

Let $\phi_r z = (z - z_1)^r / r!$ and $\psi z = (z - z_1)^q$, $q > p$. Then

$$R_f = \frac{p!}{n!} \frac{(z - z_1)^{n+q-p} f^n \zeta}{q \ldots . (q - p + 1)(\zeta' - z_1)^{q-p}}.$$

4. It is one of the properties of a holomorphic function deducible from Taylor's formula, that if such a function vanishes all along a line of finite length it vanishes throughout its holomorphic region. The limit (3) is this condition which, if fulfilled by the convergence of $S_\infty z$, secures the coincidence of fz with this series throughout this region.

The determination of the coefficients of $S_\infty z$ involves the determination of the limit of the ratio of the first minor of a Wronskian to its major when the number of rows is infinite. This is easily evaluated in two general cases; first, when the elements of the Wronskian on either side of the principal diagonal vanish; second, when the Wronskian becomes a difference-product.* It is desirable to know what other forms admit of evaluation.

5. A class of ϕ_r functions of importance, such as certain periodic functions, are holomorphic for finite values of r, but cease to be so for infinite values of r. In general, if it be merely a question of arithmetical equivalence between fz and $S_\infty z$, within a certain boundary, it will be sufficient to show that R is infinitesimal for the region considered. If we enquire further into the nature of the equivalence and the extent of the coincidence, it will be necessary to investigate the relation between the infinitesimals R and w, which, if of the orders i and j respectively, we say that the function fz and the series $S_\infty z$ have a coincidence of the order $(i - j + 1)$ with contact of the order $(i - j)$.

CHARLOTTESVILLE, *March*, 1893.

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* Annals of Mathematics, Vol. 7, No. 4, 5.

The Elliptic Inequalities in the Lunar Theory.

BY ERNEST W. BROWN, *Fellow of Christ's College, Cambridge.*

[*Continued from page* 263.]

The coefficients depending on the first power of the eccentricity having been obtained with sufficient accuracy, it is possible to proceed and obtain those depending on the square and higher powers of the same quantity. It is to be remembered that the value of c, the ratio of the two periods, was only the first term of an infinite series arranged in powers of the square of the eccentricity. Hence in obtaining the coefficients depending on higher powers of the eccentricity, it may be found necessary to have more terms of this series calculated.

V.

In the system of equations (4) put $p = 0$, there results

$$\Sigma_j\Sigma_q\{(j,\, i,\, 0,\, q)\, A_{j,\,q}A_{j-i,\,q} + (i,\, 0)\, A_{j,\,q}A_{i-j-1,\,-q} + [i,\, 0]\, A_{j,\,q}A_{-i-j-1,\,-q}\} = 0$$

for all integral values of i, except $i = 0$. Now the suffix $\pm q$ denotes that the term to which it is attached involves the eccentricity raised to the positive power q, hence in this equation only even powers of the eccentricity will occur. Omitting terms which are of an order higher than Y_0^2, we have, in accordance with the notation previously used,

$$A_{j,\,1} = \varepsilon_j, \quad A_{j,\,-1} = \varepsilon_j'.$$

We now put

$$A_{j,\,0} = a_j + \delta a_j$$

instead of a_j, where δa_j denotes the new part depending on the eccentricity; a_j has the numerical value before assigned and δa_j is evidently of the order Y_0^2.

Hence q will only take the values $0, \pm 1$. Performing these substitutions we obtain

$$\Sigma_j\{(j, i, 0, 0)(a_j + \delta a_j)(a_{j-i} + \delta a_{j-i}) + (j, i, 0, 1)\,\varepsilon_j\varepsilon_{j-i} + (j, i, 0, -1)\,\varepsilon'_j\varepsilon'_{j-i}\}$$
$$+ (i, 0)\,\Sigma_j\{(a_j + \delta a_j)(a_{i-j-1} + \delta a_{i-j-1}) + 2\varepsilon_j\varepsilon'_{i-j-1}\}$$
$$+ [i, 0]\,\Sigma_j\{(a_j + \delta a_j)(a_{-i-j-1} + \delta a_{-i-j-1}) + 2\varepsilon_j\varepsilon'_{-i-j-1}\} = 0.$$

If we omit the terms depending on Y_0^2 the equation becomes the same as that used by Dr. Hill to obtain a_i.

It is necessary to see in what way c is involved in the coefficients $(j, i, 0, 0)$, etc. Referring back to the general equations in Section I it was seen that c always occurred in one of the combinations $2i + cp$, $2j + cq$. The coefficients $(j, i, 0, 0)$, $(i, 0)$ and $[i, 0]$ will therefore be independent of c, while the two coefficients $(j, i, 0, \pm 1)$ will only involve this letter in their numerators. As the last series of coefficients only multiply quantities which are themselves of the order Y_0^2, it is sufficient for the purpose of obtaining $a_i + \delta a_i$ to this order to use the known part of the value of c which is independent of the eccentricity.

As the quantities a_i retain their previous signification, we obtain for the determination of δa_i the set of equations

$$\Sigma_j\{(j, i, 0, 0)(a_j\delta a_{j-i} + a_{j-i}\delta a_j) + (j, i, 0, 1)\,\varepsilon_j\varepsilon_{j-i} + (j, i, 0, -1)\,\varepsilon'_j\varepsilon'_{j-i}$$
$$+ 2(i, 0)(a_{i-j-1}\delta a_j + \varepsilon_j\varepsilon'_{i-j-1}) + 2\,[i, 0]\,(a_{-i-j-1}\delta a_j + \varepsilon_j\varepsilon'_{-i-j-1})\} = 0$$

omitting as before that for $i = 0$.

The labour of calculation is conducted in a way similar to that which was followed in the case of ε_i, ε'_i. The amount seems greater on account of the larger number of terms to be calculated; it is, however, not necessary to push the results to so many places of decimals. In addition, the coefficients $(j, i, 0, 0)$, $(i, 0)$ and $[i, 0]$ are given by Dr. Hill in Vol. I, pp. 245–6; his notations for them are respectively (j, i), (i) and $[i]$. The terms in ε_i, ε'_i can be calculated exactly at the first approximation by using the values previously given for these quantities, and in obtaining the second approximation it is necessary only to find the changes in δa_i. I recall that $a_i = 1$ when $i = 0$, and therefore for that value of i, $\delta a_i = 0$.

The values of δa_i are given below to eight places of decimals. The calculations were made to nine places, and to that order a third approximation was found to be unnecessary.

$$\delta a_1 = +.03938\ 170\ Y_0^2, \qquad \delta a_{-1} = +.01376\ 519\ Y_0^2,$$
$$\delta a_2 = +.00046\ 113\ Y_0^2, \qquad \delta a_{-2} = +.00002\ 216\ Y_0^2,$$
$$\delta a_3 = +.00000\ 473\ Y_0^2, \qquad \delta a_{-3} = +.00000\ 026\ Y_0^2,$$
$$\delta a_4 = +.00000\ 005\ Y_0^2, \qquad \delta a_{-4} = +.00000\ 000\ Y_0^2.$$

For the sake of completeness the value of δa_0, the part of a_0 which depends on the square of the eccentricity is obtained. We have (Vol. I, p. 131)

$$\frac{\varkappa u}{(us)^{\frac{3}{2}}} = (D^2 + 2mD)\,u + \tfrac{3}{2}m^2(u + s).$$

Substitute in this equation

$$u = (a_0 + \delta a_0)\,\Sigma_i\{a_i + \delta a_i + \varepsilon_i\zeta^c + \varepsilon_i'\zeta^{-c}\}\zeta^{2i+1},$$
$$s = (a_0 + \delta a_0)\,\Sigma_i\{a_{-i-1} + \delta a_{-i-1} + \varepsilon_{-i-1}'\zeta^c + \varepsilon_{-i-1}\zeta^{-c}\}\zeta^{2i+1},$$

where a_0 has the numerical relation to $(\mu/n^2)^{\frac{1}{3}}$ given by Dr. Hill. Since the result of such a substitution ought to be an identity for all powers of ζ, consider the set of terms independent of $\zeta^{\pm c}$. Those on the right-hand side to the order Y_0^2 are, after division by $a_0 + \delta a_0$,

$$\Sigma_i[(a_i + \delta a_i)\{(2i + 1 + m)^2 + \tfrac{1}{2}m^2\} + \tfrac{3}{2}m^2(a_{-i-1} + \delta a_{-i-1})\,\zeta^{2i+1}].$$

Put $\zeta = 1$ and let

$$H = \Sigma_i a_i\{(2i + 1 + m)^2 + 2m^2\},$$
$$\delta H = \Sigma_i \delta a_i\{(2i + 1 + m)^2 + 2m^2\}.$$

The right-hand side then becomes $H + \delta H$ when $\zeta = 1$.

In the expressions for u, s just given, put $t = t_0$, i. e. $\zeta = 1$ except where it occurs to the index $\pm c$:

$$u = (a_0 + \delta a_0)\,\Sigma_i\{a_i + \delta a_i + \varepsilon_i\zeta^c + \varepsilon_i'\zeta^{-c}\},$$
$$s = (a_0 + \delta a_0)\,\Sigma_i\{a_i + \delta a_i + \varepsilon_i'\zeta^c + \varepsilon_i\zeta^{-c}\},$$

and substitute in $\varkappa u/(a_0 + \delta a_0)(us)^{\frac{3}{2}}$. Taking out those terms as before independent of $\zeta^{\pm c}$, we obtain as far as the order Y_0^2,

$$(a_0 + \delta a_0)^2\frac{u}{(us)^{\frac{3}{2}}} = (\Sigma a_i + \Sigma\delta a_i + \Sigma\varepsilon_i\zeta^c + \Sigma\varepsilon_i'\zeta^{-c})^{-\frac{1}{2}}(\Sigma a_i + \Sigma\delta a_i + \Sigma\varepsilon_i'\zeta^c + \Sigma\varepsilon_i\zeta^{-c})^{-\frac{3}{2}}$$
$$= \frac{1}{(\Sigma a_i)^2}\left(1 - \frac{1}{2}\frac{\Sigma\delta a_i}{\Sigma a_i} - \frac{1}{2}\frac{\Sigma\varepsilon_i}{\Sigma a_i}\zeta^c - \frac{1}{2}\frac{\Sigma\varepsilon_i'}{\Sigma a_i}\zeta^{-c} + \frac{3}{4}\frac{\Sigma\varepsilon_i.\Sigma\varepsilon_i'}{(\Sigma a_i)^2}\right)$$
$$\times\left(1 - \frac{3}{2}\frac{\Sigma\delta a_i}{\Sigma a_i} - \frac{3}{2}\frac{\Sigma\varepsilon_i'}{\Sigma a_i}\zeta^c - \frac{3}{2}\frac{\Sigma\varepsilon_i}{\Sigma a_i}\zeta^{-c} + \frac{15}{4}\frac{\Sigma\varepsilon_i.\Sigma\varepsilon_i'}{(\Sigma a_i)^2}\right),$$

which gives

$$\frac{1}{(\Sigma a_i)^2}\left(1 - \frac{2\Sigma\delta a_i}{\Sigma a_i} + \frac{3}{4}\,\frac{(\Sigma\varepsilon_i + \Sigma\varepsilon_i')^2 + 4\Sigma\varepsilon_i.\Sigma\varepsilon_i'}{(\Sigma a_i)^2}\right)$$

for the terms independent of $\zeta^{\pm c}$.

Let this expression be put equal to $J + \delta J$ where $J = 1/(\Sigma a_i)^2$. Taking account of the value of $\varkappa$ which is $\mu\,(1+m)^2/n^2$, we obtain

$$a_0 + \delta a_0 = \left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}\left[\frac{(J+\delta J)(1+m)^2}{H+\delta H}\right]^{\frac{1}{3}}.$$

Dr. Hill has obtained (Vol. I, p. 144)

$$a_0 = \left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}\left[\frac{J(1+m)^2}{H}\right]^{\frac{1}{3}}$$

$$= +.99909\ 31420\left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}.$$

Hence

$$\frac{\delta a_0}{a_0} = \frac{1}{3}\cdot\frac{\delta J}{J} - \frac{1}{3}\cdot\frac{\delta H}{H}$$

$$= -\frac{2}{3}\,\frac{\Sigma\delta a_i}{\Sigma a_i} + \frac{(\Sigma\varepsilon_i + \Sigma\varepsilon_i')^2 + 4\Sigma\varepsilon_i.\Sigma\varepsilon_i'}{4\,(\Sigma a_i)^2} - \frac{\Sigma\delta a_i\,\{(2i+1+m)^2 + 2m^2\}}{3\Sigma a_i\,\{(2i+1+m)^2 + 2m^2\}}.$$

Reducing to numbers by means of the known results,

$$\frac{\delta a_0}{a_0} = -.13311\ 28\ Y_0^2,$$

and therefore the common factor for the series in u, s to the order Y_0^2 is

$$a_0 + \delta a_0 = (+.99909\ 31420 - .13299\ 21\ Y_0^2)\left(\frac{\mu}{n^2}\right)^{\frac{1}{3}}.$$

VI.

The next step forward requires us to obtain the series of coefficients $(i, \pm 2)$. In equation (4) put $p = \pm 2$ successively. It will be seen that the terms are all of the order Y_0^2 at least, and that to obtain them to this order we only need to know ε_i, ε_i' and the values of a_i; those of δa_i are not required. In this case q only takes the values 0, ± 1, ± 2, terms which involve a higher second suffix

than ± 2 being omitted. It is convenient to denote $A_{i,2}$ and $A_{i,-2}$ by f_i and f'_i respectively on the same plan as before. The equations for the determination of f_i and f'_{-i} become

$$\Sigma_j \{(j,\, i,\, 2,\, 2)\, a_{j-i} f_j + (j,\, i,\, 2,\, 0)\, a_j f'_{j-i} + (j,\, i,\, 2,\, 1)\, \varepsilon_j \varepsilon'_{j-i}$$
$$+ 2\,(i,\, 2)(a_{i-j-1} f_j + \varepsilon_{i-j-1}\varepsilon_j) + 2\,[i,\, 2](a_{-i-j-1} f'_j + \varepsilon'_{-i-j-1}\varepsilon'_j)\} = 0,$$

$$\Sigma_j \{(j-i,\, -i,\, -2,\, 0)\, a_{j-i} f_j + (j-i,\, -i,\, -2,\, -2)\, a_j f'_{j-i}$$
$$+ (j-i,\, -i,\, -2,\, -1)\, \varepsilon_j \varepsilon'_{j-i}$$
$$+ 2\,[-i,\, -2](a_{i-j-1} f_j + \varepsilon_{i-j-1}\varepsilon_j) + 2\,(-i,\, -2)(a_{-i-j-1} f'_j + \varepsilon'_{-i-j-1}\varepsilon'_j)\} = 0.$$

The first line of the second equation has been transformed by putting $j-i$ for j in order to make it uniform for calculation with the first equation. The two differ now only in their coefficients and can be dealt with together. The coefficients in the first line of each equation are transformed in the same way as in IV. It is evident that the known value of c is sufficient, all the terms being of the order Y_0^2. The terms depending on ε_i, ε'_i are fully calculated at the first approximation, and we proceed by methods like those used in obtaining the numerical values of δa_i.

The results are as follows:

$$\frac{f_0}{Y_0^2} = +.09402\ 355 \qquad \frac{f'_0}{Y_0^2} = +.03180\ 170$$

$$\frac{f_{-1}}{Y_0^2} = -.06517\ 276 \qquad \frac{f'_1}{Y_0^2} = +.01564\ 642$$

$$\frac{f_{-2}}{Y_0^2} = +.00132\ 915 \qquad \frac{f'_2}{Y_0^2} = +.00428\ 597$$

$$\frac{f_{-3}}{Y_0^2} = +.00000\ 174 \qquad \frac{f'_3}{Y_0^2} = +.00004\ 843$$

$$\frac{f_{-4}}{Y_0^2} = +.00000\ 003 \qquad \frac{f'_4}{Y_0^2} = +.00000\ 049$$

$$\frac{f_1}{Y_0^2} = +.00112\ 370 \qquad \frac{f'_{-1}}{Y_0^2} = +.00006\ 457$$

$$\frac{f_2}{Y_0^2} = +.00001\ 161 \qquad \frac{f'_{-2}}{Y_0^2} = +.00000\ 066$$

$$\frac{f_3}{Y_0^2} = +.00000\ 011 \qquad \frac{f'_{-3}}{Y_0^2} = +.00000\ 001$$

The maximum number of approximations necessary to obtain these results was four. The common divisor of f_{-1} and f'_1 is

$$(2-2c)^2\{2(2-2c)^2-2-4m+m^2\}.$$

As $c=1.07\ldots.$, the factor $(2-2c)^2$ causes this divisor to be small and f_{-1}, f'_1 correspondingly large. As f_{-1} and f'_1 are comparable in size with f_0 and f'_0, the number of approximations necessary for a given accuracy is much increased. The extra labor involved is not, however, very great, as very few of the terms in the expressions are affected.

When this divisor is expanded in powers of m, a series whose early terms (in the case of our moon) gradually increase, results. For

$$c-1=m-\frac{3}{4}m^2-\frac{201}{32}m^3-\ldots.$$

$$\frac{1}{(c-1)^2}=\frac{1}{m^2}\left(1+\frac{3}{2}m+\frac{57}{4}m^2+\ldots.\right)$$

accounting in some measure for the slow convergence of Delaunay's coefficient of the term with argument $2D-2l$ (see Section IX).

VII.

It is now necessary to determine the parts of $A_{i,\pm1}$ which depend on the cube of the eccentricity: there are none dependent on the square of that quantity, as has been pointed out. The chief difficulty here arises from the fact that it is necessary to obtain a further approximation to the value of c. This latter is, however, the principal object in view in this section, and it was necessary to push the computations thus far in order to achieve it.

In equation (4) put $p=1$ and let q take in addition such additional values that the terms depending on Y_0^3 may be included. Reference is made to Sections III and IV. Using the previous notation as far as it is available, the equation for $p=1$ is

$$\begin{aligned}P_{j,i}A_{j-i,0}A_{j,1}&+Q_{j+i,i}A_{j+i,0}A_{j,-1}+(j,i,1,2)A_{j,2}A_{j-i,1}\\&+(j,i,1,-1)A_{j,-1}A_{j-i,-2}+(i,1)(A_{i-j-1,0}A_{j,1}+A_{j,2}A_{i-j-1,-1})\\&+[i,1]\,(A_{-i-j-1,0}A_{j,-1}+A_{j,2}A_{-i-j-1,-1})=0.\end{aligned}$$

We now put as before

$$A_{i,0}=a_i+\delta a_i,\qquad A_{i,2}=f_i,\qquad A_{i,-2}=f'_i,$$

and now

$$A_{i,\,1} = \varepsilon_i + \delta\varepsilon_i, \qquad A_{i,\,-1} = \varepsilon'_i + \delta\varepsilon'_i,$$

so that ε_i, ε'_i denote the values found in the first part of this paper of the order Y_0 and $\delta\varepsilon_i$, $\delta\varepsilon'_i$ denote the new parts to be found of the order Y_0^3. Also let δc denote the new part of the ratio of the two periods of the order Y_0^2, so that $c + \delta c$ represents its full value to this order. Let us for a moment denote by F_i, the divisor

$$(2i + pc + p\delta c)\{2(2i + pc + p\delta c)^2 - 2 - 4m + m^2\}$$

which is common to all the coefficients. The coefficients (i, p), $[i, p]$ have in addition the divisor $(2i + pc + p\delta c)$. Multiply the equation by F_i; it will then be free from fractions involving $c + \delta c$ in the denominators except as regards the coefficients just mentioned. The equation has been already solved when δc, $\delta\varepsilon_i$, $\delta\varepsilon'_i$, δa_i and terms of higher order are put equal to zero. Hence taking the variations of c, ε_i, ε'_i, a_i and including such terms of higher orders as are required, we have, after division by F_i (in which expression δc can be put zero), for $p = 1$

$$\begin{aligned}
&\{P_{j,\,i}a_{j-i} + 2\,(i,\,1)\,a_{i-j-1}\}\delta\varepsilon_j + \{Q_{j+i,\,i}a_{j+i} + 2\,[i,\,1]\,a_{-i-j-1}\}\delta\varepsilon'_j \\
&+ \{P_{j,\,i}\delta a_{j-i} + 2\,(i,\,1)\,\delta a_{i-j-1}\}\varepsilon_j + \{Q_{j+i,\,i}\delta a_{j+i} + 2\,[i,\,1]\,\delta a_{-i-j-1}\}\varepsilon'_j \\
&+ \left[a_{j-i}\frac{\partial}{\partial c}(P_{j,\,i}F_i) + 2a_{i-j-1}\frac{\partial}{\partial c}\{(i,\,1)\,F_i\}\right]\frac{\varepsilon_j\delta c}{F_i} \\
&\qquad\qquad + \left[a_{j+i}\frac{\partial}{\partial c}(Q_{j+i,\,i}F_i) + 2a_{-i-j-1}\frac{\partial}{\partial c}\{[i,\,1]\,F_i\}\right]\frac{\varepsilon'_j\delta c}{F_i} \\
&+ (j,\,i,\,1,\,2)f_j\varepsilon_{j-i} + (j,\,i,\,1,\,-1)\,\varepsilon'_j f'_{j-i} + 2\,(i,\,1)f_j\varepsilon'_{i-j-1} + 2\,[i,\,1]\varepsilon_j f'_{-i-j-1} = 0.
\end{aligned}$$

Here Σ_j is to be understood to stand before each term. There will be a similar equation for $p = -1$, $\delta\varepsilon_i$ and $\delta\varepsilon'_{-i}$ being a pair which are calculated together.

The method of proceeding is by continued approximation as in the determination of ε_i, ε'_i. We obtain $\delta\varepsilon_i$, $\delta\varepsilon'_i$ for a sufficient number of values of i, *excluding* $i = 0$, in terms of $\delta\varepsilon_0$, $\delta\varepsilon'_0$, $Y_0\delta c$, Y_0^3, the results as is evident being linear functions of these quantities with numerical coefficients. The results are then substituted in the two equations given by $i = 0$ (or two equivalent equations) and we obtain two linear relations between the same four quantities.

The number of terms to be computed is very large, but the following obser-

vations will show that the amount of calculation is really smaller than at first sight appears.

There are four sets of terms involving in combination (i) the letters a, $\delta\epsilon$; (ii) the letters ϵ, δa; (iii) a, ϵ, δc; (iv) ϵ, f. These can be dealt with separately at the first approximation.

In set (i) we are going to determine the parts of $\delta\epsilon_i$, $\delta\epsilon'_i$ which depend on $\delta\epsilon_0$, $\delta\epsilon'_0$. But the expressions are exactly the same as those given in IV for determining ϵ_i, ϵ'_i in terms of ϵ_0, ϵ'_0. The results are therefore to hand.

In set (ii) the coefficients are the same as those in IV and are therefore known; also as δa_i, ϵ_i, ϵ'_i are known, the set can be completely calculated at the first approximation in terms of Y_0^3.

In set (iii) the new coefficients have to be found, a_i, ϵ_i, ϵ'_i are known. These terms can also be completely obtained at the first approximation, expressed in terms of $Y_0\delta c$; as δc comes out approximately .0027 Y_0^3, these terms are calculated to two fewer places of decimals than those involving Y_0^3.

Set (iv) can be calculated completely at the first approximation in terms of Y_0^3, as all the quantities are known. In addition the coefficients $(i, 1)$, $[i, 1]$ have been found, while as

$$(j, i, 1, q) = (2j + qc) L_i + (2j + qc)^2 M_i,$$

and that L_i, M_i have been calculated, these coefficients are soon obtained.

Having calculated as far as possible all the terms at the first approximation, we use the values of $\delta\epsilon_i$, $\delta\epsilon'_i$ thus obtained for the second approximation. Set (i) will be the only one affected, and its coefficients have been found. These approximations will of course add to the coefficients of Y_0^3 and $Y_0\delta c$, but not to those of $\delta\epsilon_0$ and $\delta\epsilon'_0$, as is evident from the remarks on set (i).

The coefficients ϵ_i, ϵ'_i have been expressed in terms of Y_0 or $\epsilon_0 - \epsilon'_0$. It now appears that Y_0 has an additional part $\delta\epsilon_0 - \delta\epsilon'_0$ to be added to it. This really amounts to introducing a new arbitrary constant, or, in other words, to making $\epsilon_0 - \epsilon'_0 + \delta\epsilon_0 - \delta\epsilon'_0$ the arbitrary constant instead of $\epsilon_0 - \epsilon'_0$. This is inconvenient and causes unnecessary complications. Since we are at liberty to choose our arbitrary, it will be simplest to make it Y_0, so that $\delta Y_0 = 0$. Hence the coefficient of the principal elliptic term in the expression for the coordinate y will, in this theory, be

$$a_0 + \delta a_0 Y_0,$$

where $a_0 + \delta a_0$ has the value which is found above expressed in terms of $(\mu/n^2)^{\frac{1}{3}}$ and Y_0^2. We have then $\delta\epsilon_0' = \delta\epsilon_0$.

The values of $\delta\epsilon_i$, $\delta\epsilon_i'$ found as above are given below.

$$
\begin{array}{llll}
\delta\epsilon_{-1} = & +.02435\ 517\ Y_0^3 & +.22567\ 789\ \delta\epsilon_0 & +.76306\ 1\ Y_0\delta c, \\
\delta\epsilon_1' = & -.00877\ 697\ Y_0^3 & -.08834\ 235\ \delta\epsilon_0 & -.24185\ 2\ Y_0\delta c, \\
\delta\epsilon_1 = & +.02529\ 357\ Y_0^3 & +.00215\ 229\ \delta\epsilon_0 & -.00006\ 1\ Y_0\delta c, \\
\delta\epsilon_{-1}' = & +.00322\ 634\ Y_0^3 & -.00128\ 260\ \delta\epsilon_0 & +.00079\ 8\ Y_0\delta c, \\
\delta\epsilon_{-2} = & +.00070\ 947\ Y_0^3 & +.00008\ 499\ \delta\epsilon_0 & +.00003\ 7\ Y_0\delta c, \\
\delta\epsilon_2' = & +.00526\ 427\ Y_0^3 & -.00049\ 147\ \delta\epsilon_0 & -.00145\ 3\ Y_0\delta c, \\
\delta\epsilon_2 = & +.00048\ 131\ Y_0^3 & +.00000\ 620\ \delta\epsilon_0 & -.00000\ 1\ Y_0\delta c, \\
\delta\epsilon_{-2}' = & +.00002\ 534\ Y_0^3 & -.00000\ 076\ \delta\epsilon_0, & \\
\delta\epsilon_{-3} = & +.00000\ 493\ Y_0^3 & -.00000\ 019\ \delta\epsilon_0, & \\
\delta\epsilon_3' = & +.00009\ 895\ Y_0^3 & -.00000\ 333\ \delta\epsilon_0 & -.00001\ 0\ Y_0\delta c, \\
\delta\epsilon_3 = & +.00000\ 678\ Y_0^3 & -.00000\ 001\ \delta\epsilon_0, & \\
\delta\epsilon_{-3}' = & +.00000\ 031\ Y_0^3, & & \\
\delta\epsilon_{-4} = & +.00000\ 006\ Y_0^3, & & \\
\delta\epsilon_4' = & +.00000\ 137\ Y_0^3. & &
\end{array}
$$

In order to obtain the numerical ratios of $\delta\epsilon_0$, δc to Y_0^3, we may use the two equations of the system just considered for $i = 0$. As will be stated below, an equation of verification was, however, computed,—in this case, the second of equations (6) in the first part of this paper, with the necessary terms of order Y_0^3 included. These three equations gave respectively

$$
\begin{array}{l}
+.00425\ 226\ Y_0^3 + \ .02946\ 89\ \delta\epsilon_0 - 1.11898\ 7\ Y_0\delta c = 0 \\
+.00902\ 093\ Y_0^3 + \ .00129\ 30\ \delta\epsilon_0 - 3.33860\ 6\ Y_0\delta c = 0 \\
+.18016\ 812\ Y_0^3 + 4.32005\ 66\ \delta\epsilon_0 + \ .98781\ 5\ Y_0\delta c = 0.
\end{array}
$$

The first of these combined with the third gave

$$\delta c = +.00268\ 561\ Y_0^3,$$

and the second with the third gave 2 instead of 1 for the last place of decimals. Whence

$$\delta\epsilon_0 = \delta\epsilon_0' = -.04231\ 912\ Y_0^3.$$

Substituting in the table just given for the values of $\delta\varepsilon_i$, we obtain as the final values for these

$$\frac{\delta\varepsilon_{-1}}{Y_0^3} = +.01685\ 40 \quad \frac{\delta\varepsilon'_1}{Y_0^3} = -.00568\ 79 \quad \frac{\delta\varepsilon_1}{Y_0^3} = +.02520\ 23 \quad \frac{\delta\varepsilon'_{-1}}{Y_0^3} = +.00328\ 28$$

$$\frac{\delta\varepsilon_{-2}}{Y_0^3} = +.00070\ 60 \quad \frac{\delta\varepsilon'_2}{Y_0^3} = +.00528\ 12 \quad \frac{\delta\varepsilon_2}{Y_0^3} = +.00048\ 10 \quad \frac{\delta\varepsilon'_{-2}}{Y_0^3} = +.00002\ 54$$

$$\frac{\delta\varepsilon_{-3}}{Y_0^3} = +.00000\ 49 \quad \frac{\delta\varepsilon'_3}{Y_0^3} = +.00009\ 91 \quad \frac{\delta\varepsilon_3}{Y_0^3} = +.00000\ 68 \quad \frac{\delta\varepsilon'_{-3}}{Y_0^3} = +.00000\ 03$$

$$\frac{\delta\varepsilon'_4}{Y_0^3} = +.00000\ 14$$

It is not my intention to carry the calculation of the coefficients to any further extent. Enough has been given to show how they can be obtained to any desired accuracy; and the methods outlined above for making the computations can be arranged so that the assistance of a computer of ordinary intelligence can be largely made use of. One of the chief advantages of these methods is the ease with which equations of verification can be employed. Before comparing some of the results with the corresponding ones of Delaunay, I shall give a short account of the methods of verification used.

VIII.

Verification of the Results.

There appear to be two principal sources of error in the class of computations here treated apart from that produced by the use of logarithms. The first is the common one of making an actual numerical mistake of any kind whatever; the second arises from a term omitted altogether. It is therefore necessary to have such equations of verification as shall give some security that errors proceeding from *both* these sources may be eliminated. Owing to the interdependence of the various approximations and, in the case of the higher coefficients, to the large number of terms to be computed, it is advisable to have at each step a test for errors arising from the first and chief source. For this object the equation most useful, on account of its simplicity, is the second of equations (3). It was used in all the calculations for coefficients depending on the square and cube of the eccentricity. In it the terms are arranged in the same way as those in the equations used for finding the results; that is to say, the suffixes proceed in

the same order, and the coefficients depending on m, c alone differ. An error therefore of a sensible term omitted altogether in finding any pair of coefficients would probably also occur in the equation of verification. Such a case occurred once in these computations. For the coefficients depending on the first power of the eccentricity, one of equations (6) was used, as the computations of the coefficients of its various terms were simple, and as it served at one calculation to verify the four quantities of the same order, ε_i, ε'_{-i}, ε'_i, ε_{-i}, while the second of equations (3) required two calculations, one for ε_i, ε'_{-i} and another for ε'_i, ε_{-i}. It will be noticed, too, that the terms in equations (6) are arranged somewhat differently, a property which possesses some value in this connection.

To detect an error of the second kind, reliance was placed on the computation of an expression which has an entirely different form. In the equation

$$uD^2s + sD^2u + Du.Ds - 2m(uDs - sDu) + \frac{9m^2}{4}(u+s)^2 = C$$

substitute the values of u, s, Du, etc., found and arrange the result in powers of Y_0 and $\zeta^{\pm c}$. The result will be of the form

$$\frac{C}{(a_0+\delta a_0)^2} = \alpha + \alpha' Y_0^2 + (\beta Y_0 + \beta' Y_0^3)(\zeta^c + \zeta^{-c}) + \gamma Y_0^2(\zeta^{2c} + \zeta^{-2c}) + \dots .$$

Here α, α', β, β', γ will be series in ζ^{2i} with known numerical coefficients. As this equation should be true for *all* values of ζ, we should have

$$\frac{C}{(a_0+\delta a_0)^2} = \alpha + \alpha' Y_0^2, \quad \beta Y_0 + \beta' Y_0^3 = 0, \quad \gamma = 0$$

to the degree of approximation used; moreover, since Y_0 is arbitrary, in addition we have

$$\beta = 0, \quad \beta' = 0.$$

Again, as these equations ought to hold for all values of ζ, we can put $\zeta = 1$ in each of them; the verification depends on the results being identically zero. Its value arises from the fact that the expressions to be here calculated are all of such forms as

$$\Sigma_i a_i, \quad \Sigma(2i+1)a_i, \quad \Sigma(2i+1+c)^2\varepsilon_i, \quad \text{etc.,}$$

they are therefore totally different from any of those previously used. The higher coefficients as $\delta\epsilon_3$, f_4, etc., are, however, multiplied by large numbers in some cases, and therefore unless these are carried to a higher degree of approximation than was before found necessary, this method fails to test quite fully the results obtained. For practical purposes, since the coefficients have been computed to a large number of places of decimals, the test appears to be sufficient.

The Jacobian constant is obtained by putting $p = 0$ in the first of equations (3). Let C now denote the initial value of this constant as found by Dr. Hill, and δC the new part depending on Y_0^2. From the equation we obtain by means of the known values of δa_i,

$$\frac{\delta C}{a_0^2} = + .34552\ 75\ \ Y_0^2.$$

The value of δC obtained from the equation of verification above where ζ is put equal to unity, differs only by one unit in the last place. This therefore gives the required verification for the series of coefficients δa_i. The coefficients f_i, f_i' are absent from these expressions.

The case of these coefficients δa_i requires special mention. The two values of δC found from two entirely independent expressions differ only by one unit in the seventh figure. That is to say, they are the same within the limits of error. It seems very improbable that an error of one unit in the fourth significant figure of δa_1 or δa_{-1}, or of the first significant figure in δa_2 or δa_{-2}, should have been able to elude detection while running through two separate sets of equations of verification. Yet this is roughly the amount of error required to make the results agree with those of Delaunay as shown in the next section.

In order to verify the transformations from rectangular to polar coordinates, in which the method of 'special values' was adopted, an extra 'special value' was computed and the results were found to be accordant.

All computations once made were gone through a second time. The average error made in the later portions of the work was about one in every four or five hundred figures. It did not seem to be confined to any particular class of operation. In using the new eight-figure tables of the French Government, extra care was exercised for the differences, and the chance of error thus diminished.

IX.

Transformation to Polar Coordinates.

Let v denote the difference between the true and mean longitudes. Gathering together the results of the previous sections and adding those given by Dr. Hill (Vol. I, p. 248), we have with Delaunay's notation for the arguments; that is, $2D$ for that of the 'Variation' and l for that of the principal elliptic term:

$$\begin{aligned}
\frac{r\cos v}{a_0+\delta a_0} = 1 &+ (-.00718\ 00395 + .05314\ 689\ \ Y_0^2)\cos 2D \\
&+ (+.00000\ 60424 + .00048\ 328\ \ Y_0^2)\cos 4D \\
&+ (+.00000\ 00325 + .00000\ 499\ \ Y_0^2)\cos 6D \\
&+ (+.00000\ 00002 + .00000\ 005\ \ Y_0^2)\cos 8D \\[6pt]
&+ (-.49679\ 1802\ \ Y_0 - .08463\ 82\ \ Y_0^3)\cos l \\
&+ (-.09332\ 9284\ \ Y_0 + .01116\ 60\ \ Y_0^3)\cos (2D - l) \\
&+ (+.00025\ 6338\ \ Y_0 + .00598\ 71\ \ Y_0^3)\cos (4D - l) \\
&+ (+.00000\ 2210\ \ Y_0 + .00010\ 40\ \ Y_0^3)\cos (6D - l) \\
&+ (+.00000\ 0025\ \ Y_0 + .00000\ 14\ \ Y_0^3)\cos (8D - l) \\
&+ (+.00134\ 2824\ \ Y_0 + .02848\ 51\ \ Y_0^3)\cos (2D + l) \\
&+ (+.00001\ 0769\ \ Y_0 + .00050\ 64\ \ Y_0^3)\cos (4D + l) \\
&+ (+.00000\ 0079\ \ Y_0 + .00000\ 71\ \ Y_0^3)\cos (6D + l) \\[6pt]
&+ .12582\ 524\ \ Y_0^2\cos 2l \\
&- .04952\ 634\ \ Y_0^2\cos (2D - 2l) + .00118\ 827\ \ Y_0^2\cos (2D + 2l) \\
&+ .00561\ 512\ \ Y_0^2\cos (4D - 2l) + .00001\ 227\ \ Y_0^2\cos (4D + 2l) \\
&+ .00005\ 017\ \ Y_0^2\cos (6D - 2l) + .00000\ 012\ \ Y_0^2\cos (6D + 2l) \\
&+ .00000\ 051\ \ Y_0^2\cos (8D - 2l).
\end{aligned}$$

$$\begin{aligned}
\frac{r\sin v}{a_0+\delta a_0} = &+ (+.01021\ 14544 + .02561\ 651\ \ Y_0^2)\sin 2D \\
&+ (+.00000\ 57149 + .00043\ 897\ \ Y_0^2)\sin 4D \\
&+ (+.00000\ 00276 + .00000\ 447\ \ Y_0^2)\sin 6D \\
&+ (+.00000\ 00002 + .00000\ 005\ \ Y_0^2)\sin 8D
\end{aligned}$$

$+(+1.00000\ 0000\ Y_0 \qquad\qquad\qquad)\sin l$

$+(+\ .20446\ 5776\ Y_0 - .02254\ 19\ Y_0^3)\sin(2D-l)$

$+(+\ .00036\ 0509\ Y_0 + .00457\ 52\ Y_0^3)\sin(4D-l)$

$+(+\ .00000\ 1960\ Y_0 + .00009\ 41\ Y_0^3)\sin(6D-l)$

$+(+\ .00000\ 0023\ Y_0 + .00000\ 13\ Y_0^3)\sin(8D-l)$

$+(+\ .00159\ 6237\ Y_0 + .02191\ 96\ Y_0^3)\sin(2D+l)$

$+(+\ .00000\ 9426\ Y_0 + .00045\ 57\ Y_0^3)\sin(4D+l)$

$+(+\ .00000\ 0069\ Y_0 + .00000\ 65\ Y_0^3)\sin(6D+l)$

$+ .06222\ 185\ Y_0^2 \sin 2l$

$+ .08081\ 918\ Y_0^2 \sin(2D-2l) + .00105\ 914\ Y_0^2 \sin(2D+2l)$

$+ .00295\ 682\ Y_0^2 \sin(4D-2l) + .00001\ 094\ Y_0^2 \sin(4D+2l)$

$+ .00004\ 669\ Y_0^2 \sin(6D-2l) + .00000\ 011\ Y_0^2 \sin(6D+2l)$

$+ .00000\ 046\ Y_0^2 \sin(8D-2l).$

To find the longitude we can proceed as follows. The difference of the true and mean longitudes is denoted by

$$v = \tan v - \tfrac{1}{3}\tan^3 v + \tfrac{1}{5}\tan^5 v - \ldots .$$

Tan v is obtained from the above values and the terms are expanded in powers of Y_0. To Y_0^0 are attached the sines of the arguments $2iD$, to Y_0^1 the sines of $2iD \pm l$, and to Y_0^2 those of $2iD$, $2iD \pm 2l$. The coefficients depending on Y_0^1 have already been obtained and discussed in the first part of this paper. As far as the order Y_0^2 let

$$r\cos v = S_0 + S_1 Y_0 + S_2 Y_0^2$$
$$r\sin v = S_0' + S_1' Y_0 + S_2' Y_0^2$$

where S_0, S_1, etc., denote the terms involving the respective powers of Y_0, etc., in the expressions just given. It is not difficult to prove that when these are substituted in the expansion for v, we obtain for the terms involving Y_0^2 in v

$$\frac{S_0S_2' - S_0'S_2}{S_0^2 + S_0'^2} + \frac{S_0S_0'(S_1^2 - S_1'^2) - S_1S_1'(S_0^2 - S_0'^2)}{(S_0^2 + S_0'^2)^2}.$$

The arguments $\pm 2l$ only appear in the numerators. The coefficients in this expression of $\cos 2l$, $\sin 2l$, and that in which $2l$ does not appear, are quickly

separated out and the results are series involving sines and cosines of the arguments $2iD$. To each of these three sets of terms the method of special values for $2D$ can then be applied and the required coefficients of the terms in longitude obtained.

The results for these terms are:

$$
\begin{aligned}
+ Y_0^2 [&+ .12015\ 93 \sin 2D + .30993\ 73 \sin 2l \qquad\qquad + .08545\ 32 \sin(2D - 2l) \\
&+ .00209\ 70 \sin 4D + .00532\ 16 \sin(2D + 2l) + .01248\ 18 \sin(4D - 2l) \\
&+ .00002\ 70 \sin 6D + .00006\ 81 \sin(4D + 2l) + .00021\ 37 \sin(6D - 2l) \\
&+ .00000\ 03 \sin 8D + .00000\ 07 \sin(6D + 2l) + .00000\ 27 \sin(8D - 2l)].
\end{aligned}
$$

The relation between Y_0 and Delaunay's e has been found in Section IV. By means of it these coefficients expressed in seconds of arc become

$$
\begin{aligned}
&+ 298''.959 \sin 2D + 771''.132 \sin 2l \qquad\qquad + 212''.610 \sin(2D - 2l) \\
&+ \quad 5''.217 \sin 4D + \ 13''.240 \sin(2D + 2l) + \ 31''.055 \sin(4D - 2l) \\
&+ \quad 0''.067 \sin 6D + \quad 0''.169 \sin(4D + 2l) + \quad 0''.532 \sin(6D - 2l) \\
&+ \quad 0''.001 \sin 8D + \quad 0''.002 \sin(6D + 2l) + \quad 0''.007 \sin(8D - 2l).
\end{aligned}
$$

The part of the motion of the Lunar Perigee which depends on Y_0^2 has been found above to be

$$\delta c = + .00268\ 561\ Y_0^2;$$

and the relation between Y_0 and Delaunay's eccentricity,

$$Y_0 = 2e \times 1.00027\ 136.$$

Also

$$\delta\left(\frac{1}{n}\cdot\frac{d\omega}{dt}\right) = -\frac{1}{1+m}\,\delta c;$$

hence

$$\delta\left(\frac{1}{n}\cdot\frac{d\omega}{dt}\right) = -.00994\ 29\ e^2.$$

The series given by Delaunay for this quantity is*

$$
\begin{aligned}
-e^2\Big[\frac{3}{8} m_1^2 + \frac{675}{64} m_1^3 + \frac{31605}{512} m_1^4 + \frac{1483665}{4096} m_1^5& \\
+ \frac{25291729}{16384} m_1^6 - \frac{352038885}{1179648} m_1^7&\Big],
\end{aligned}
$$

*Comptes Rendus, Tom. LXXIV, p. 19.

where $m_1 = n'/n$. In numbers the portion within the brackets becomes

$$.00209\ 82 + .00441\ 42 + .00193\ 25 + .00084\ 82 + .00027\ 04 - .00000\ 39 = .00955\ 96,$$

which is considerably less than the previous value, the difference being greater than the last term but one of this series. Moreover, Delaunay's series seems to indicate that when estimation is made for the uncalculated portion, his value should be even less still. The results, therefore, can scarcely be considered accordant. The difference amounts to $0''.055$ in the motion.

The coefficients found above agree with what might have been expected from Delaunay's series, with one exception, the part of the coefficient of the variation which depends on e^2. Expressed in seconds of arc, Delaunay's values for the terms e^2m, e^2m^2, etc., in the coefficient are*

$$217''.976 + 59''.839 + 16''.331 + 3''.852 + 0''.777 + 0''.137 - 0''.049 = 298''.863;$$

and it would appear from this that the true value was somewhat less, say

	$298''.84$
the value obtained above is	$298''.96$
a difference of	$0''.12$

It should be pointed out that in Prof. Newcomb's comparison of Delaunay's and Hansen's Lunar Theories,† there seems no reason to expect that the former's value of the coefficient is erroneous by so much as one-tenth of a second. It may, however, have happened that there is an error in some other part of the long expression for this coefficient which may balance a possible error here, or, what is more likely, that the other terms if carried further would have given some considerable coefficient. The terms in e'^2 converge very slowly up to the last given.

It may be mentioned here, that in comparing the results for many of the coefficients with those given by Delaunay the differences are less than a hasty inspection would have caused us to estimate. If we examine the series arranged in powers of m, more carefully, and especially those which have a power of the

* Mém. de l'Acad. des Sc., Tom. XXIX, p. 815.

† Astron. Papers of Amer. Eph., Vol. I, p. 92.

eccentricity as a factor, it appears that in most cases the terms have a tendency sooner or later to change sign and the term before the change of sign has a comparatively small numerical multiplier. Delaunay sometimes stops at this term, thus giving a false idea of the accuracy of the total result.

The most serious difference is that in the value of the part of the Lunar Perigee just obtained, depending on e^2 and the ratio of the mean motions only, since, as Dr. Hill has pointed out,* the motion of the Perigee is capable of being found observationally with a high degree of accuracy. It is seen from the expansion reproduced above that Delaunay in this very slowly converging series has carried his terms beyond the change of sign, and another change of sign in the uncalculated portion, which would be necessary to make the two results accordant, seems hardly likely to occur early enough to produce the difference. In either theory the difference, just mentioned, in the coefficient of sin $2D$ would be sufficient to cause the discordance. As I have said in Section VIII, it seems hardly probable that a sufficient error has been made in the calculations undertaken to obtain δa_i. It might be surmised that the error arises from the set $\delta\epsilon_i$, $\delta\epsilon_i'$ The error required there would be several units in the third place of decimals, and again this seems improbable in view of the two sets of equations of verification.

One other coefficient is worthy of special notice, namely that of sin $(2D-2l)$. Delaunay gets for this†

$$130''.786 + 46''.089 + 22''.283 + 8''.131 + 3''.179 + 1''.308 + 0''.542$$
$$= 212''.318$$

my coefficient being $\quad 212''.610$

so that the remainder of Delaunay's series is $+0''.29$. The values of f_i, f_i' not depending on those of δa_i and vice versâ, an error in one set does not involve one in the other.

In a similar way may the parts depending on Y_0^3 be transformed to polar coordinates. The coefficient of sin l in longitude will be of the form

$$\alpha Y_0 + \beta Y_0^3$$

---

* On the Part of the Motion of the Lunar Perigee, etc. Acta Math., Vol. VIII, p. 1.

† Mem. cit., p. 823.

where α, β are numerical coefficients. In Delaunay's theory this coefficient is

$$2e - \tfrac{1}{4}e^3,$$

and therefore the second approximation to Y_0 in terms of e will be given by

$$\alpha Y_0 = 2e - \tfrac{1}{4}e^3 - \frac{8\beta}{\alpha^3}e^3.$$

This must be applied to all terms with arguments $2iD \pm l$ when it is desired to express them in Delaunay's notation.

HAVERFORD COLLEGE, *June*, 1893.

On the Multiplication of Semi-convergent Series.

By Florian Cajori.

In vol. 24, p. 44 of Math. Ann., A. Voss has given the necessary and sufficient conditions which must be satisfied in order that Cauchy's rule for the multiplication of series be applicable to semi-convergent series Σa_n and Σb_n, in case that one of them, say Σa_n, becomes absolutely convergent when expressed in the form $\Sigma(a_{2n} + a_{2n+1})$. The purpose of the present article is to extend Voss's results.

It is always possible to find a series converging toward the product of the sums of two semi-convergent series, when one of the factor-series can be made absolutely convergent on associating its terms into groups, each containing a finite number of terms. Thus, if $U_n = \sum_0^n a_n$ and $V_n = \sum_0^n b_n$ are semi-convergent, and if $U_n = \sum_0^n g_n$ is absolutely convergent (g_n being the $(n+1)^{\text{th}}$ group in the first series), then by a theorem of Mertens,

$$UV = \sum_0^\infty (b_n g_0 + b_{n-1} g_1 + \ldots . + b_0 g_n).$$

If the product $\sum_0^n (a_0 b_n + a_1 b_{n-1} + \ldots . + a_n b_0)$, of $\sum_0^n a_n$ and $\sum_0^n b_n$, formed according to Cauchy's rule, is convergent, then by a theorem of Abel, it converges to UV. From this it follows that the necessary and sufficient condition for the convergence of the product-series is that

$$\sum_0^\infty (b_n g_0 + b_{n-1} g_1 + \ldots . + b_0 g_n) = \sum_0^\infty (a_0 b_n + a_1 b_{n-1} + \ldots . + a_n b_0) \qquad \text{(I)}$$

Let us suppose that all the groups contain the same number p of terms, so that $g_n = a_{pn} + a_{pn+1} + \dots + a_{pn+p-1}$, then for the case $n = pm$,

$$
\begin{aligned}
\sum_0^n (b_n g_0 + b_{n-1} g_1 + \dots + b_0 g_n) - \sum_0^n (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0) \\
= b_0 \{a_{pm+1} + a_{pm+2} + \dots + a_{p^2m+p-1}\} \\
+ b_1 \{a_{pm} + a_{pm+1} + \dots + a_{p^2m-1}\} + \dots + b_{pm} \{a_1 + a_2 + \dots + a_{p-1}\} \\
= b_0 (a_{pm+1} + \dots + a_{pm+p-1}) \\
+ \sum_{i=0}^{i=m-1} \{b_{pi+2} a_{pm-pi-1} + b_{pi+3} (a_{pm-pi-2} + a_{pm-pi-1}) + \dots \\
+ b_{pi+p} (a_{pm-pi-p+1} + a_{pm-pi-p+2} + \dots + a_{pm-pi-1})\} + E.
\end{aligned}
$$

If $pm = 2ps$, then

$$
\begin{aligned}
E \equiv\ & g_1 \{b_{2ps-1} + b_{2ps-2} + \dots + b_{2ps-p+1}\} \\
& + g_2 \{b_{2ps-2} + b_{2ps-3} + \dots + b_{2ps-2p+1}\} + \dots \\
& + g_s \{b_{2ps-s} + b_{2ps-s-1} + \dots + b_{ps+1}\} \\
& + g_{s+1} \{b_{2ps-s-1} + b_{2ps-s-2} + \dots + b_{ps-p+1}\} + \dots + g_{2ps-1} \{b_1 + b_0\} + g_{2ps} b_0.
\end{aligned}
$$

If $pm = 2ps + 1$, then

$$
\begin{aligned}
E \equiv\ & g_1 \{b_{2ps} + b_{2ps-1} + \dots + b_{2ps-p+2}\} \\
& + g_2 \{b_{2ps-1} + b_{2ps-2} + \dots + b_{2ps-2p+2}\} + \dots \\
& + g_s \{b_{2ps-s+1} + b_{2ps-s} + \dots + b_{ps+2}\} \\
& + g_{s+1} \{b_{2ps-s} + b_{2ps-s-1} + \dots + b_{ps-p+2}\} + \dots + g_{2ps} \{b_1 + b_0\} + g_{2ps+1} b_0.
\end{aligned}
$$

In either case a quantity β and an infinitesimal quantity, ϵ_s, approaching zero as s increases indefinitely, can be so chosen that, for large values of s (letting $|x|$ stand for the absolute value of x),

$$
|E| < \epsilon_s \{|g_1| + |g_2| + \dots + |g_s|\} + \beta \{|g_{s+1}| + |g_{s+2}| + \dots + |g_{pm}|\}.
$$

Since Σg_n is absolutely convergent, it follows that the second member of the inequality approaches the limit zero as s increases indefinitely. Therefore E approaches zero as a limit, and the condition that equation (I) be satisfied for $n = pm$, is

$$
\begin{aligned}
\operatorname*{Lim}_{m=\infty} \sum_{i=0}^{i=m-1} \{b_{pi+2} a_{pm-pi-1} + b_{pi+3} (a_{pm-pi-2} + a_{pm-pi-1}) + \dots \\
+ b_{pi+p} (a_{pm-pi-p+1} + a_{pm-pi-p+2} + \dots + a_{pm-pi-1})\} = 0.
\end{aligned}
$$

By a similar process of reasoning we obtain the condition that (I) be satisfied, for the case $n = pm + r$, viz.

$$\underset{m=\infty}{Lim}\sum_{i=0}^{i=m-1}\{b_{pi+r+2}a_{pm-pi-1} + b_{pi+r+3}(a_{pm-pi-2} + a_{pm-pi-1}) + \dots \\ + b_{pi+p+r}(a_{pm-pi-p+1} + \dots + a_{pm-pi-1})\} = 0. \quad \text{(II)}$$

If we agree to let r represent successively all integral values from 0 to $p - 1$ (both inclusive), then *expression* (II) *embodies p equations which constitute together the necessary and sufficient conditions for the existence of equation* (I) *and for the applicability of Cauchy's multiplication rule to* Σa_n and Σb_n.

Another set of necessary and sufficient conditions can be deduced from conditions (II), viz.

Cauchy's multiplication rule is applicable to Σa_n and Σb_n, if the n^{th} term of the product-series always approaches the limit zero and if ONE *of the p conditions in* (II) *is satisfied.*

We first prove that if the n^{th} term of the product-series approaches zero as n increases indefinitely and if the $(r + 1)^{\text{th}}$ condition in (II) is satisfied, then the r^{th} condition in (II) is also satisfied. We have (disregarding, as we may, a finite number of terms $a_{n-x}b_x$ in which $x < r + 2$ or $x > pm + r$),

$$\underset{m=\infty}{Lim}\left[\sum_{i=0}^{i=m-1}\{b_{pi+r+2}a_{pm-ip-1} + b_{pi+r+3}(a_{pm-pi-1} + a_{pm-pi-2}) + \dots \\ + b_{pi+p+r}(a_{pm-pi-1} + \dots + a_{pm-ip-p+1})\} \\ + \sum_{i=0}^{i=m} g_{m-i}(b_{pi+r+1} + b_{pi+r+2} + \dots + b_{pi+p+r-1}) \\ - \sum_{r=r}^{r=p+r-2}(a_{pm+r+1}b_0 + a_{pm+r}b_1 + \dots + a_0 b_{pm+r+1})\right]$$

$$= \underset{m=\infty}{Lim}\sum_{i=0}^{i=m}\{b_{pi+r+1}a_{pm+p-pi-1} + b_{pi+r+2}(a_{pm+p-pi-1} + a_{pm+p-pi-2}) + \dots \\ + b_{pi+p+r-1}(a_{pm+p-ip-1} + \dots + a_{pm-ip+1})\}.$$

In the first member of this equation we assume the $(r + 1)^{\text{th}}$ condition of (II) to be satisfied and the sum of $p - 1$ successive terms in the product-series to

approach zero. Remembering that Σg_n is absolutely convergent, it will be seen that the first member of the equation approaches the limit zero; hence the second member approaches zero. If in the second member we put $m-1$ in place of m, then the expression assumes the form of the r^{th} condition in (II). Hence if the $(r+1)^{\text{th}}$ condition is satisfied, the r^{th} condition is satisfied. But if the r^{th} condition is satisfied, then the $(r-1)^{\text{th}}$ is satisfied, etc., and the theorem is established.

A set of *sufficient* conditions of convergence is obtained by taking the absolute values of all the a's and all the b's in (II). In the light of the conditions thus obtained one may readily see the correctness of a condition established by A. Pringsheim (Math. Annalen, Vol. 21, p. 334), which asserts that the product-series is convergent and Cauchy's rule applicable, if $\sum\limits_0^n a_n b_n$ is absolutely convergent and remains so when any number of factors a_m, b_m is replaced by other factors of higher indices. Pringsheim proves that when his condition is satisfied, then the n^{th} term of the product-series approaches zero. Moreover, if $\sum\limits_0^n a_n b_n$ converges absolutely, then not only is $\sum\limits_n^\infty a_n b_n = 0$, but zero is also the limit of the sum of all the terms $a_x b_y$ in the expression, after making substitutions of the kind above referred to, provided that $x + y \gtreqless 2n$. But the values of the indices x and y may be so chosen that the absolute values of all the products ab involved in any one of the p conditions in expression (II) will be included. Hence Pringsheim's *sufficient* condition is true, for whenever that condition is satisfied, then the *necessary* and *sufficient* conditions developed in this paper are satisfied. But observe that we have verified Pringsheim's condition only for the case that Σa_n becomes absolutely convergent on associating its terms into groups containing an equal number p of terms, while his condition is also applicable to cases in which the number of terms in the various groups is not the same.

As an example, consider the product of the two semi-convergent series

$$\sum_1^m \left(\frac{1}{3m} + \frac{1}{3m+1} - \frac{2}{3m+2}\right) \text{ and } \sum_1^m \left\{\frac{(-1)^m}{\log 3m} + \frac{(-1)^m}{\log(3m+1)} + \frac{(-1)^m}{\log 3m+2)}\right\}.$$

The first of these becomes absolutely convergent when $p = 3$. Pringsheim's test fails here, for $\sum_0^n a_n b_n$ is not absolutely convergent. Applying the second test developed in this paper, we observe that the n^{th} term of the product-series approaches the limit zero as n increases indefinitely, no matter whether n equals $3m$, $3m+1$ or $3m+2$. The first condition in (II) is also satisfied, for

$$\begin{aligned} \underset{m=\infty}{Lim} \Big[\Big\{ &+ \frac{1}{\log 5} \cdot \frac{2}{3m-4} - \frac{1}{\log 8} \cdot \frac{2}{3m-7} + \frac{1}{\log 11} \cdot \frac{2}{3m-10} \\ &- \ldots . \pm \frac{1}{\log(3m-4)} \cdot \frac{2}{5} \Big\} \\ + \Big\{ &- \frac{1}{\log 3} \left(\frac{1}{3m-2} - \frac{2}{3m-1} \right) + \frac{1}{\log 6} \left(\frac{1}{3m-5} - \frac{2}{3m-4} \right) \\ &- \ldots . \pm \frac{1}{\log(3m-3)} \left(\frac{1}{4} - \frac{2}{5} \right) \Big\} \Big] = 0. \end{aligned}$$

Hence the product-series converges and Cauchy's multiplication rule is applicable.

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On Certain Ruled Surfaces of the Fourth Order.

By Thomas F. Holgate.

Introduction.

The Ruled Surface of the Fourth Order has been studied to a considerable extent both by analytical and by synthetic processes. Perhaps the most important and complete analytical treatment of the surface is by Cayley in his second and third "Memoirs on Skew Surfaces, otherwise Scrolls" (Philosophical Transactions, 1864 and 1869, respectively). In these the surface is considered from a purely algebraic point of view, and the author is concerned chiefly with the determination of the possible different varieties of the surface. These are distinguished by the nature of the double curve, and eight species are enumerated in the second memoir, to which two others are added in the third memoir.

The principal synthetic treatment of the surface is by Cremona (Sulle superficie gobbe di quarto grado, Mem. della R. Istoria di Bologna, Serie II, T. VIII), who makes his first distinction between those surfaces which are of deficiency zero and those of deficiency unity. Of the former he finds ten species, of the latter, two, thus obtaining two surfaces not observed by Cayley. This enumeration of twelve species is, I believe, generally conceded to be complete. The surface has also been studied to a greater or less extent by Rohn, Salmon, Chasles, Reye and others.

In the following paper I consider those species of the surface which may be generated by two projective sheaves of planes of the second order. These all admit a trinodal quartic section, and are consequently of deficiency zero. A direct demonstration was necessary at the outset that two such sheaves of planes generate in general a Ruled Surface of the Fourth Order, and after this follows a discussion of the different species in what appears to be the most natural order of succession. The reciprocal of each of these surfaces and the bitangent torse

accompanying each are pointed out, and an enumeration of the subforms of the different species is made which appears to be complete. In order to avoid frequent references it was deemed advisable to preface the treatment of the Ruled Surface with a brief statement of those properties of the line congruences of which constant use is made in the subsequent work.

§1.—*The Line Congruence of the First Order.*

1. Let there be given two collinear bundles of rays and planes, designated by their centres S and S', which are neither concentric nor in perspective position, and which, in general, have no self-corresponding ray or plane. Corresponding forms in these bundles—for example, sheaves of lines which lie in corresponding planes, sheaves of planes whose axes are corresponding rays, corresponding cones or sheaves of planes of the second order—are projectively related to one another.* The line system, Σ, generated by the intersection of corresponding planes in the bundles S and S', is a congruence of the first order, since through every point in space there passes one, and in general but one, line of this congruence.† If in any point P two lines of the congruence intersect, then is P a "singular point" with respect to the congruence, or more briefly, a singular point of the congruence. For in this point must also a pair of corresponding rays of the bundles intersect, and consequently an infinite number of lines of the congruence which lie on a cone of the second order, and which together pass through all possible singular points of the congruence.‡

If α and α' be any two corresponding planes in the bundles S and S', the sheaves of lines of the first order lying in these planes, whose centres are the centres of the bundles, are projective to one another, and hence the line of intersection of these planes contains two projective ranges of points which lie in one another. These have two self-corresponding points which may be either real and distinct, coincident or imaginary.|| These self-corresponding points are singular points of the congruence, since in them corresponding rays of the bundles intersect. Every line of the system Σ, therefore, contains two singular points, and conversely, every straight line which passes through two singular points is a line of the congruence, since it is the intersection of a pair of corresponding planes in the bundles. No straight line contains more than two singular points,

* Reye, Geometrie der Lage, Ab. II, S. 5.
† Reye, G. d. L., II, S. 84.
‡ Reye, G. d. L., II, S. 85.
|| Reye, G. d. L., I, S. 169.

for then would every point of this line be a singular point, and the bundles would contain at least one self-corresponding plane, namely, the planes determined by the ray SS' and its corresponding ray in each of the bundles would coincide, and would consequently be self-corresponding, which is contrary to supposition.

It is readily seen that singular points of the congruence always exist, since the ray SS' of the bundle S must meet its corresponding ray of S' in the point S', and hence S' is a singular point. Similarly, S is a singular point of the congruence. The lines of the system Σ which pass through S lie, therefore, on a cone of the second order, K, which passes through all singular points of the congruence, in particular through the point S'. Similarly, the lines of the system which pass through S' lie on a cone of the second order, K', which passes through all singular points of the congruence, in particular through the point S. Hence all the singular points of the congruence lie on the curve of intersection of the cones K and K'. This curve, which we shall denote by k^3, passes through both the points S and S', and is of the third order, since an arbitrary plane intercepts it in at least one, but in general and at most in three points.* Every point of k^3 is a singular point of the congruence, since in each point of this curve two rays of the cones, i. e. two lines of the congruence intersect. The points S and S' are wholly arbitrary points of k^3,† and therefore out of any two points of this curve the lines of the congruence are projected by two collinear bundles, and the curve itself, by two projective cones of the second order whose rays are lines of the congruence.

If γ be any plane through S and γ' its corresponding plane through S', these will intersect in a line of the congruence which will contain two singular points, i. e. will intersect k^3 in two points, which, however, are real and distinct, coincident or imaginary, according as γ and γ' contain two real and distinct, two coincident or two imaginary rays of the cones K and K' respectively. Every line of the congruence Σ may therefore be considered a chord of the curve k^3. Those chords which meet k^3 in two real and distinct points we shall denote as "actual chords"; those which meet k^3 in two consecutive points, as "tangents," and those which meet the curve in two imaginary points, as "ideal chords" of this curve. No straight line can meet k^3 in more than two points, and a line which meets k^3 in a single point does not belong to this congruence.

*Reye, G. d. L., II, S. 88.

†Reye, G. d. L., II, S. 90.

Since those planes which contain two consecutive rays of a cone of the second order form a sheaf of planes of the second order,* it follows immediately that the tangent system of the twisted cubic curve k^3 is projected out of any two points of the curve, S and S', by means of two projective sheaves of planes of the second order.

It has been already observed that an arbitrary plane cuts k^3, in general, and at most in three points, and it is evident that if these three points are all real and distinct, the plane will contain three actual chords of the curve. If two of these points approach indefinitely near to each other and finally coincide, i. e. if the plane meet k^3 in two points at one of which it touches the curve, two of the three chords lying in the plane will coincide and the third will become a tangent at the point of contact of the plane. If all three points of intersection coincide, i. e. if the plane osculate the curve at a certain point, then the three chords will fall together and will coincide with the tangent at the point of osculation. Finally, two of the points of intersection may become imaginary, in which case the plane will contain but one real chord, and that an ideal chord.

Thus the bundles S and S' generate in the most general case a line congruence of the first order and third class, whose singular points lie on a non-degenerate twisted cubic curve passing through the centres of the bundles.

2. Suppose now that the collinear bundles S and S' have one self-corresponding plane, α, but no self-corresponding ray. The projective sheaves of lines lying in α generate a curve of the second order, k^2, passing through S and S', each of whose points is a singular point of the congruence arising from these bundles. The remaining singular points lie in a straight line, k, which meets the singular conic, but which does not lie in a plane with it.† This singular line, k, is itself a line of the congruence, being the intersection of a pair of corresponding planes of S and S'.

Each line of the congruence meets both k^2 and k, since every plane of the bundles S and S' must intercept the conic in some second point, and must also intercept the straight line. Conversely, every straight line which meets both k^2 and k is a line of the congruence. Those lines of the congruence which pass through any point of k^2 lie in a plane with k; in particular, the lines of the congruence which pass through O, the point of intersection of k^2 and k, lie in that

* Reye, G. d. L., I, S. 87.

† Reye, G. d. L., II, S. 85.

plane through k which is tangent to k^2. The lines of the congruence which pass through any point of the singular line k lie on a cone of the second order of which k is always one ray, and which projects the conic k^2; only for the point O does this cone break up into two planes, namely, the self-corresponding plane α and the plane through k tangent to k^2.

Out of any point of k the congruence is projected by a sheaf of planes of the first order, each plane of the sheaf containing an infinite number of lines of the system, while out of any two points of the conic k^2 the congruence is projected by means of two collinear bundles,* so that S and S' may be any points whatever of k^2, the point O being the only exceptional point.

Through every point in space there passes one, and in general but one, line of this congruence, while an arbitrary plane contains in general two of these lines, namely, the lines joining the point of intersection of the plane with k to its two points of intersection with the conic k^2. Thus the congruence generated by the collinear bundles S and S' which have a self-corresponding plane, but no other self-corresponding element, is of the first order and second class, its singular points lying on a conic and a straight line which meet but which do not lie in the same plane.

3. If the two bundles S and S' be so situated as to have the ray joining their centres a self-corresponding ray, then must every plane through this ray correspond again to a plane through this ray. In general, there will be two self-corresponding planes, α_1 and α_2, passing through the ray SS', which, however, may be either real and distinct, coincident or imaginary.

Consider, first, the self-corresponding planes, α_1 and α_2, to be real and distinct. The pairs of projective sheaves of lines lying in these planes generate straight lines, k_1 and k_2, respectively, every point of which is a singular point of the congruence, and beside the points of these lines there is no singular point outside SS', for if so, there must be a third self-corresponding plane, in which case the bundles S and S' would be in perspective position† contrary to supposition.

Every line of the congruence generated by S and S' meets both singular lines k_1 and k_2, for any pair of corresponding planes must pass through the same points of these lines; and conversely, every line which meets both these singu-

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* Reye, G. d. L., II, S. 86.

† Reye, G. d. L., II, S. 15.

lar lines, the ray SS' included, is a line of the congruence. Those lines belonging to this congruence which pass through any point of k_1 lie therefore in a plane with k_2; and similarly, the lines of the congruence which pass through any point of k_2 lie in a plane with k_1. Out of any point, therefore, of either k_1 or k_2 the congruence is projected by means of a sheaf of planes of the first order, while out of any two points in the line SS' it is projected by two collinear bundles, of which SS' is a self-corresponding ray. The centres of the bundles generating the congruence may, consequently, be any two points of this ray, excepting only the points O_1 and O_2 in which the ray intersects k_1 and k_2 respectively. Out of these two points the congruence is projected in the same manner as out of any other points of the singular lines.

The two singular lines k_1 and k_2 intersect SS', but not in the same point, for then would all lines of the congruence lie in one plane and the bundles would again be in perspective position.

The congruence is as before of the first order, but in this case, of the first class, since an arbitrary plane contains only one line of the congruence, namely, the line joining the points in which k_1 and k_2 respectively intersect this plane.

If now the self-corresponding planes α_1 and α_2 approach indefinitely near to each other and finally coincide in a single plane α_{12}, the lines of singular points k_1 and k_2 will at the same time approach indefinitely near to each other, and will finally coincide in the line k_{12} which lies in α_{12}. Every point of k_{12} is a singular point of the congruence, and besides the points of this line there is no singular point outside SS'. Every line of the congruence meets k_{12}, while those which pass through any one point of the singular line lie in a plane with this line, since they must lie in a plane which passes through all the singular points of the system. The points of the singular line k_{12}, and the planes through this line, are projectively related to one another, each point corresponding to that plane in which lie the lines of the congruence passing through it, for the range of points and the sheaf of planes are both perspective to every regulus of the second order which belongs to the congruence.

Finally, if the self-corresponding planes become imaginary, no real singular point exists outside the line SS', while out of any two points of this line the congruence is projected by means of two collinear bundles as before.

§2.—*The most General Ruled Surface of the Fourth Order.*

4. In the two collinear bundles S and S', respectively, which in the most general case generate a line congruence of the first order and third class, whose singular curve is a non-degenerate twisted cubic k^3, select any pair of corresponding sheaves of planes of the second order, Φ_1 and Φ_1'. These are related projectively to one another and generate by the intersections of corresponding planes a ruled surface, which shall be denoted by F_1^4, each generator of which is a chord of k^3. This ruled surface is of the fourth order, i. e. an arbitrary straight line meets in general four generators of this surface, and if any straight line meet more than four generators it must meet every generator and lie wholly on the surface.

Let p be an arbitrary straight line and π an arbitrary plane through it, cutting the sheaves Φ_1 and Φ_1' in two projective sheaves of lines of the second order, Π and Π'. The points where corresponding rays of Π and Π' intersect are the points in which the plane π cuts the generators of F_1^4. We shall first show that either at most four of these points of intersection, or else an infinite number of them, lie on the arbitrary straight line p.

5. Let a, b, c, d, e be any five rays of the sheaf Π, and a', b', c', d', e' be the corresponding rays of the sheaf Π'. Suppose that the points of intersection A, B, C, D, E of these five pairs of corresponding rays all lie in the straight line p. Project the two sheaves of lines, Π and Π', out of any point O not lying in the plane π, by means of two projective and concentric sheaves of planes of the second order, Ω and Ω', and out of the same point project the line p by means of the plane π_1. The planes $\alpha, \beta, \gamma, \delta, \varepsilon$ of the sheaf Ω, in which lie the lines a, b, c, d, e of the sheaf Π, will intersect the corresponding planes $\alpha', \beta', \gamma', \delta', \varepsilon'$ of the sheaf Ω' in rays a_1, b_1, c_1, d_1, e_1 which lie in the plane π_1 and pass through the point O. Construct a regulus of the second order perspective to the sheaf Ω.* This is cut by the plane π_1 in a conic which is perspective to the sheaf Ω, and consequently projectively related to the sheaf Ω'. But since, by supposition, five pairs of corresponding planes of the two sheaves Ω and Ω' intersect in rays lying in the plane π_1, five points of the conic must lie in the planes corresponding to them of the sheaf Ω', to which it is projective, and therefore the conic is perspective also to the sheaf Ω'.† Hence all pairs of

* Reye, G. d. L., I, S. 132.

† Reye, G. d. L., I, S. 136.

corresponding planes in the two sheaves intersect in lines which lie in the plane π_1 and pass through the point O, since each pair has in common the point O and a point of the conic. In the plane π_1, then, lie an infinite number of these lines of intersection, and consequently in the line p lie an infinite number of points of intersection of corresponding rays of the two sheaves Π and Π'. The line p, therefore, meets an infinite number, i. e. all of the generators of the surface F_1^4, if in any case it meet more than four of them.

6. Next, let r be an arbitrary straight line not meeting the twisted cubic k^3, and not a chord of k^3. Project the line r out of S by means of the plane ϕ, and find ϕ' the corresponding plane of the bundle S'. Also project r out of S' by means of the plane θ', and find θ the corresponding plane of the bundle S. Then ϕ and ϕ' intersect in a chord of k^3 which meets the arbitrary line r, as do also θ and θ'. In r choose at will three points L, M, N. Project these out of S by the rays l, m, n, and find in the bundle S' the corresponding rays l', m', n'. Similarly, project L, M, N out of S' by the rays l', m', n', and find in the bundle S the corresponding rays l_1, m_1, n_1. The plane λ of the bundle S, determined by the rays l and l_1, corresponds to the plane λ' of the bundle S', determined by the rays l' and l_1'. So also the plane μ, determined by m and m_1, corresponds to the plane μ', determined by m' and m_1', and the plane ν, determined by n and n_1, corresponds to the plane ν' determined by n' and n_1'. The corresponding planes λ and λ' intersect in a chord of k^3 which meets the line r, since both planes pass through the point L of r. Similarly μ and μ', and likewise ν and ν', intersect in chords of k^3 which meet the arbitrary straight line r.

Thus we have obtained five planes in each bundle, viz. ϕ, θ, λ, μ, ν in S, and ϕ', θ', λ', μ', ν' in S', which correspond two and two, corresponding pairs intersecting in chords of k^3 which meet the arbitrary straight line r. These two sets of five planes each determine in the bundles S and S' two corresponding and hence projective sheaves of planes of the second order, Φ_2 and Φ_2', such that the lines of intersection of five pairs of corresponding planes intercept the straight line r. But we have shown that if any straight line meet more than four such lines of intersection, it must meet every line of intersection of pairs of corresponding planes of the two sheaves. Moreover, through every point of r passes a pair of corresponding planes of these sheaves. For, since r lies in the plane ϕ and also in the plane θ', it is cut by the planes of the two sheaves Φ_2 and Φ_2' in

two projective ranges of points which lie in one another, and have three self-corresponding points, namely, the points L, M and N. Hence every point of the line is a self-corresponding point, i. e. through every point of r goes a pair of corresponding planes of Φ_2 and Φ_2'. Since no two chords of the twisted cubic k^3 can intersect in a point of r, every chord of k^3 which meets the arbitrary line r must lie in a pair of corresponding planes Φ_2 and Φ_2'. Therefore

All the chords of a twisted cubic curve which meet an arbitrary straight line having no point in common with the cubic, are projected out of any two points of the curve by means of projective sheaves of planes of the second order.

The arbitrarily chosen sheaves Φ_1 and Φ_1' may have, and in general will have, four planes in common with the sheaves Φ_2 and Φ_2' respectively without coinciding with them. Hence the ruled surface F_1^4 may have, and in general will have, four generators which meet the arbitrary straight line r.

Thus we have shown that the ruled surface which is generated by two projective sheaves of planes of the second order is such that an arbitrary straight line may, and in general will, meet four of its generators, and if any straight line meet more than four generators it must meet every generator, and lies wholly on the surface. The surface is therefore of the fourth order.

7. The sheaves of planes Φ_1 and Φ_1' which generate the surface F_1^4 envelope cones of the second order, which we shall denote by K_1 and K_1, respectively, whose vertices are points of the twisted cubic k^3. Each generator of F_1^4 is tangent to both the cones K_1 and K_1', but since either of these cones is determined as soon as the sheaf which envelopes the other is chosen, the surface F_1^4 is completely defined as "that surface whose generators are chords of a twisted cubic k^3, and tangents to a cone of the second order K_1 whose vertex is a point of the cubic." The surface thus defined we shall consider to be the most general Ruled Surface of the Fourth Order.

8. Since the system of tangents to the curve k^3 is projected out of S and S' by means of two projective sheaves of planes of the second order, it forms the system of generators of a ruled surface of the fourth order F^4, which, however, is a developable surface or torse, since each generator intersects its adjacent generators in points of the twisted cubic. The sheaves Φ and Φ' which generate the torse F^4 envelope the cones K and K' which project the points of the cubic out of S and S' respectively.

9. The projective sheaves of planes Φ_1 and Φ'_1 which are chosen arbitrarily in the bundles S and S', have in general, and at most four planes in common with the sheaves Φ and Φ' respectively, which project the tangents to k^3, and consequently the surface F_1^4 has in general four generators which are tangents to k^3. Also the cones K_1 and K'_1 have in general, and at most, four rays in common with the cones K and K' respectively. If any ray of K_1 coincide with a ray of K, its corresponding ray in K'_1 will coincide with the corresponding ray of K', and hence these rays will intersect in a point of k^3. If any ray of K_1 lie either within or without K, its corresponding ray in K'_1 will lie within or without K', but these corresponding rays will not intersect.

All planes of Φ_1 and Φ'_1 intersect the twisted cubic in the points S and S', respectively, but beside this, if any ray of K_1 lie within the cone K, the plane of Φ_1 through this ray will contain two rays of K, and consequently will meet the cubic k^3 in two points different from S. The corresponding plane of Φ'_1 will pass through these same two points, and hence these planes will intersect in a generator of the surface F_1^4 which is an actual chord of k^3. If the sheaves Φ_1 and Φ'_1 be so chosen that the cones K_1 and K'_1 enveloped by them lie wholly within the cones K and K' respectively, every generator of the surface F_1^4 will be an actual chord of the twisted cubic.

The plane of Φ_1 passing through a ray of K_1 which coincides with a ray of K in general contains a second ray of K, and hence will intersect the corresponding plane of Φ'_1 in a generator of F_1^4 which is an actual chord of k^3. The two rays of K which lie in this plane may, however, coincide, as when the two cones are tangent to each other along this ray, in which case the generator becomes a tangent to k^3.

If the sheaf of planes Φ_1 be so chosen that some of the rays of the cone K_1 lie outside the cone K, while others lie inside this cone, there must be at least two rays of K_1, say a and b, which coincide with rays of K. There will be at the same time at least two planes of the sheaf Φ_1, say μ and ν, which coincide with planes of Φ. These planes μ and ν separate the planes of Φ_1 into two groups, in one of which lie the planes through the rays a and b. Every plane of this group will contain two rays of the cone K, and consequently will, with its corresponding plane in Φ'_1, give rise to a generator of F_1^4 which is an actual chord of the twisted cubic. Among the planes of the other group there may, and in general will, arise two other planes which are common to the sheaves Φ_1 and Φ.

If these do not appear, the planes of this group will, with their corresponding planes in Φ_1', give rise to generators of F_1^4 which are ideal chords of k^3, since they meet the cubic only in the point S.

If the four common planes of Φ_1 and Φ all appear, they divide the planes of the sheaf Φ_1 into four groups; the planes in two of these groups, with their corresponding planes in Φ_1', give rise to generators which are actual chords of k^3, while the planes of the remaining two groups give rise to generators which are ideal chords of the cubic. The groups which give rise to actual-chord generators alternate between the groups which give rise to ideal-chord generators. Consequently the generators themselves are in general arranged in four groups, of which the four tangent generators already referred to form the boundaries, the two groups of actual-chord generators alternating between the two groups of ideal-chord generators.

10. On the other hand, since the cone K on which the twisted cubic lies has in general four rays in common with the cone K_1, the curve k^3 will in general meet the cone K_1 in four points different from S, and will lie partly within and partly without this cone. Any point of k^3 which lies outside the cone K_1 lies also outside the corresponding cone K_1', and similar conclusions are true for points of the curve which lie either on or inside the cone K_1. Through every point of k^3 which lies outside the cone K_1 will pass two real distinct planes of the sheaf Φ_1, and likewise the corresponding two planes of the sheaf Φ_1', so that, through every point of k^3 which lies outside the cone K_1 will pass two real, distinct generators of the surface F_1^4. Through those points of k^3 which lie inside K_1 will pass no real plane of Φ_1, and consequently no real generator of F_1^4; while through each of the points in which k^3 meets the cone pass two consecutive planes of Φ_1, and consequently two consecutive generators of F_1^4. In other words, through any point of k^3 which lies outside K_1 will pass two real, distinct generators of the surface F_1^4. As this point moves along the curve toward the cone, the two generators tend toward each other and finally coincide for the point of the curve which lies on the cone; as the point moves inside the cone, the generators through it become imaginary. The twisted cubic is thus a double curve on the surface, and may lie either *actually* or *ideally* upon the surface.

11. Since the tangent plane to the surface at any point contains the generator through that point, and since the tangent plane at any point along the

twisted cubic contains the tangent to the cubic at that point, there are at each point of the twisted cubic two tangent planes to the surface, namely, those planes determined by the tangent to the cubic and each of the two generators which meet the cubic at this point. If one of the generators at a point of the cubic coincide with the tangent at this point, the tangent plane through this generator becomes the osculating plane to the cubic, while the other tangent plane in general remains distinct; that is, one of the tangent planes to the surface at a point where a tangent generator meets the double cubic is the osculating plane to the cubic at that point.

At those points of the double curve in which the two generators coincide, namely, at those points of k^3 which lie on the cones K_1 and K_1', the two tangent planes to the surface must also coincide. Such points have been designated by Professor Cayley as *pinch points*, so that the ruled surface of the fourth order F_1^4 has in general four pinch points. These are distributed along the double curve in such a manner as to separate the segments of the curve which lie actually upon the surface from those segments which lie ideally upon the surface. For convenience, these latter segments may be referred to as the *isolated* segments of the double curve. Thus, suppose A, B, C, D be the four points taken in order along the double cubic in which this curve meets the cones K_1 and K_1', and that the segment AB of the curve lies outside the cones. Then the segment BC will lie inside the cones, CD outside, and DA again inside. Through every point of the segments AB and CD will pass two real, distinct generators of the surface; these segments will lie actually upon the surface, and at each point of them there will be two real and distinct tangent planes to the surface. The segments BC and DA of the double curve are isolated segments, and the tangent planes at the points of these segments are imaginary. Through each of the boundary points, A, B, C, D, there pass two consecutive generators, and consequently the two tangent planes at these points coincide. These are, therefore, the four pinch points on the surface.

The four tangent generators which in general appear evidently meet k^3 in those segments which lie actually upon the surface, two in each segment. Denote these generators by r, s, t, v, and suppose that r and s meet k^3 in the segment AB, while t and v meet the curve in the segment CD. If, now, the pinch points A and B move toward each other along the curve and finally coincide at a point P, that is, if the cones K_1 and K become tangent to each other

along the ray SP, and likewise the cones K_1' and K' become tangent to each other along the ray $S'P$, then the two tangent generators r and s fall together and become a single tangent generator at the pinch point P; and generally, whenever two pinch points fall together, the generator through this point becomes a tangent generator.

12. As we consider the distribution of pinch points and tangent generators along the double curve, that is, as we consider the possible ways in which the sheaves Φ_1 and Φ_1', and the cones K_1 and K_1' enveloped by them, may be chosen as related to the cones K and K', several different varieties or subforms of the general surface F_1^4 arise. These may be conveniently arranged as follows:

I.—*Subforms in which the Four Pinch Points are all Real.*

(1). The pinch points A, B, C, D are all distinct; the tangent generators r, s, t, v appear on the surface and are also distinct.

In this case the cones K_1 and K_1' have four rays in common with the cones K and K', respectively, namely, those rays which project the points A, B, C, D out of S and S', respectively. The sheaves Φ_1 and Φ_1' likewise have four planes in common with the sheaves Φ and Φ', respectively, these giving rise to the tangent generators r, s, t, v. The cubic curve k^3 lies actually upon the surface throughout two segments, AB and CD, but the remaining two segments are isolated. There are two groups of actual-chord generators and two groups of ideal-chord generators. Each actual-chord generator meets the cubic twice in the same segment.

(2). The pinch points A and B coincide at a point P; the tangent generators r and s fall together and become a single tangent generator at P. The remaining two pinch points C and D, and the tangent generators t and v, exist and remain distinct.

The cones K and K_1 have two distinct rays in common, and are tangent to each other along a third ray SP, while the sheaves Φ and Φ_1 have three planes in common, one of which passes through the ray SP. Similar relations exist between the corresponding cones and sheaves in the bundle S'. The cubic curve lies actually on the surface only throughout the segment CD and at the point P. One group of actual-chord generators reduces to the single tangent generator at the pinch point P.

(3). The pinch points A and B coincide at a point P; also C and D coincide at a point Q. The tangent generators r and s fall together and pass through P, while t and v also fall together and pass through Q.

This case arises when the cone K lies within the cone K_1, the two being tangent to each other along the two rays SP and SQ. The cubic curve is wholly isolated except at the pinch points P and Q. The two groups of actual-chord generators reduce to the single tangent generators through the pinch points P and Q, respectively, all other generators being ideal chords of k^3.

(4). The pinch points B and C coincide at a point Q; the tangent generators s and t also fall together and become a single tangent generator at Q. The remaining pinch points A and D, and the tangent generators r and v, exist and remain distinct.

The cubic curve lies actually on the surface throughout the double segment AQD. One group of ideal-chord generators disappears, and the two groups of actual-chord generators are only separated by the tangent generator through the pinch point Q. One tangent generator, r, meets the cubic in a point of the segment AQ, while the remaining one, v, meets the cubic in the segment QD. Each actual-chord generator meets the cubic twice in the same segment.

(5). The four pinch points are all distinct, but the tangent generators all disappear or are imaginary.

The cones K and K_1 have four common rays, but the sheaves Φ and Φ_1 have no planes in common. The cubic curve lies actually on the surface in two segments, AB and CD, and is isolated in the remaining two segments. Every generator is an actual chord of the cubic, and meets the cubic once in a point of the segment AB and once in a point of the segment CD.

(6). Two pinch points, B and C, of subform (5) coincide at a point Q, at which point there arises a tangent generator. The remaining two pinch points, A and D, are distinct, but there are no other real tangent generators.

The cones K and K_1 are tangent to each other along the ray SQ, and intersect along the rays SA and SD. The sheaves Φ and Φ_1 have only one plane in common, namely, the plane through the common ray SQ. The cubic curve lies actually on the surface throughout the double segment AQD. Every gene-

rator of the surface, beside the tangent generator, is an actual chord of the cubic, and meets this curve once in the segment AQ and once in the segment QD.

(7). The pinch points B and C coincide at a point P, while D and A coincide at a point Q. Two tangent generators appear on the surface and pass through the pinch points P and Q, respectively.

In this case the cones K and K_1 are tangent to each other along the rays SP and SQ, while through these rays pass planes common to the two sheaves Φ and Φ_1. The cubic curve lies actually on the surface throughout, and all generators besides the two tangent generators are actual chords of the cubic.

This variety admits of a subdivision according as it arises from the coincidence of pairs of pinch points in subform (1) or in subform (5); that is, according as a generator meets the double curve twice in the same segment PQ, or once in each segment.

(8). Three of the pinch points, A, B and C, coincide at a point P, while the fourth, D, remains distinct. The three tangent generators, r, s and t, fall together and become a single tangent generator through the pinch point P; the fourth tangent generator, v, must appear on the surface and remain distinct.

The cones K and K_1 have two rays SP and SD in common, and have contact of the second order along SP. The sheaves Φ and Φ_1 likewise have two distinct planes in common, of which one is the plane through the common ray SP. The cubic curve lies actually on the surface throughout one of its segments PD, but is isolated in the other segment. There will be one group of actual-chord generators and one group of ideal-chord generators, these groups being separated by the two tangent generators.

(9) and (10). The four pinch points coincide at a point P; the four tangent generators also coincide and pass through P.

The cones K and K_1 have only one ray in common, the ray SP, but have contact of the third order along this ray. The sheaves Φ and Φ_1 have a common plane through SP. Two cases arise according as the cone K_1 lies inside K, or K inside K_1. In the first case (9) the cubic curve lies actually on the surface throughout, and every generator beside the tangent generator at P is an actual chord of k^3. In the second case (10) the cubic curve lies actually on the sur-

face only at the pinch point P, while all generators except the tangent generator are ideal chords of the cubic.

II.—*Subforms in which Two of the Pinch Points are Real and Two Imaginary.*

(11). Any two adjacent pinch points, A and B, and the corresponding tangent generators, r and s, disappear or become imaginary, while the remaining two pinch points, C and D, and the remaining tangent generators, t and v, appear on the surface and are distinct.

The cones K and K_1 have two rays in common, the sheaves Φ and Φ_1 have two planes in common. The cubic curve lies actually on the surface at every point of the one segment CD, and is isolated throughout the other segment. There is one group of actual-chord generators in which the two tangent generators appear, and one group of ideal-chord generators.

(12) and (13). Two pinch points and two tangent generators disappear; the remaining two pinch points coincide at a point P, while the remaining two tangent generators fall together and become a single tangent generator at the pinch point P.

The cones K and K_1 have only one ray in common, namely, the ray SP, along which they are tangent to each other. The sheaves Φ and Φ_1 have but one common plane, the plane through the ray SP. Two different cases arise again according as the cone K_1 lies within the cone K, or the cone K within the cone K_1. In the first case (12) the cubic curve lies actually on the surface throughout, and all generators are actual chords except the tangent generator at the pinch point. In the second case (13) the cubic lies actually on the surface only at the pinch point P, and all generators except the tangent generator are ideal chords of the cubic.

(14). Two pinch points coincide at a point P, the remaining two pinch points disappear. Two tangent generators fall together and pass through the pinch point P, the remaining two appear on the surface and are distinct.

Each of the cones K and K_1 lies outside the other, but they are tangent to one another along the ray SP. The cubic lies actually on the surface throughout, but does not meet all the generators. There is one group of actual-chord

generators in which the tangent generator through the pinch point appears, and one group of ideal-chord generators, these groups being separated by the two remaining tangent generators.

III.—*Subforms in which the Four Pinch Points are all Imaginary.*

(15). The four pinch points disappear, but the four tangent generators all exist on the surface and are distinct.

The cones K and K_1 have no ray in common, and each lies wholly outside the other. Of the sheaves Φ and Φ_1, however, there are four common planes which, together with their corresponding planes in Φ' and Φ_1', give rise to the four tangent generators. The cubic curve lies wholly on the surface, but does not meet all the generators. There are two groups of actual-chord generators and two groups of ideal-chord generators occurring alternately, each group of the one sort being separated from the two groups of the other sort by two tangent generators. The two generators which pass through any one point of the cubic always belong to different groups of actual-chord generators.

(16) and (17). The four tangent generators as well as the four pinch points, disappear.

Neither the cones K and K_1, nor the sheaves of planes Φ and Φ_1, have elements in common. Of the cones K and K_1, the one lies wholly inside the other. Two subforms of the surface arise according as K_1 lies inside K, or K inside K_1. In the first case (16) the double curve lies actually on the surface at every point, and all generators are actual chords of the cubic. In the second case (17) the double curve is isolated throughout, and all generators are ideal chords of the cubic.

13. It has already been shown that an arbitrary straight line meets in general, and at most, four generators of the surface F_1^4, consequently an arbitrary plane α cuts the surface in a curve which possesses this same property of being met by an arbitrary straight line of its plane generally and at most in four points. Since α intercepts the double curve of the surface in general in three points, through each of which two generators pass, the curve of section in general has three double points; that is, the surface F_1^4 is cut by an arbitrary plane in general in a curve of the fourth order with three double points.

But this curve is generated by the two projective sheaves of lines of the second order in which the sheaves of planes Φ_1 and Φ_1' are cut by the arbitrary plane α. Hence *two projective sheaves of lines of the second order which lie in the same plane, and which in general have no self-corresponding ray, generate a curve of the fourth order which has three double points.*

14. If the plane of section α contain one generator of the surface, the two projective sheaves of lines in which Φ_1 and Φ_1' are cut will have this generator as a self-corresponding ray, and will generate, in addition to this straight line, a curve which is intercepted by this generator as well as every other straight line of the plane generally and at most in three points. This curve must have one double point, since the plane α cuts the double curve of the surface in one point other than the intersections of the generator with the double curve; that is, any section of the surface F_1^4 by a plane through one generator consists of this generator and a curve of the third order with a double point.

If the generator p through which the plane is passed be an actual chord of the double curve k^3, in each point of intersection of this generator with k^3 the plane will intercept a second generator, and consequently the curve of section will always pass through these two points. These points remain fixed for all planes through the generator. The third point of intersection of the generator with the curve of section is that point at which the plane cuts the two consecutive generators. The plane is said to be tangent to the surface at this point, for it not only contains the generator through the point but also a second line which here meets the surface in three consecutive points. Every plane through a generator is tangent to the surface at some point along this generator, since at some point it cuts the two consecutive generators; the point of tangency varies along the generator with each different plane of section.

When the point of tangency of the plane coincides with either of the intersections of the generator p with k^3, this point must be the double point on the cubic curve of section, since at this point the tangent plane cuts not only the generators consecutive to p, but also the second generator through this point of the double curve. The tangent to the double curve at this point also lies in this plane, as has already been stated.

If the actual-chord generator lying in a plane of section pass through one of the four pinch points of the surface, the curve of section must be tangent to

this generator at the pinch point. In this point the plane cuts the generators consecutive to the given generator, and both on the same side, while the double point on the curve of section exists elsewhere. In a certain sense every plane through this generator is tangent to the surface at the pinch point, for at this point every plane cuts the consecutive generators. The curve of section of the plane containing the generator and the tangent to k^3 at the pinch point consists of a cubic with a cusp at the pinch point, since for this plane the double point of the cubic falls into the pinch point. This is the true tangent plane at this point.

If, however, a plane be passed through an ideal-chord generator, the curve of section will meet this generator only in the point at which the plane is tangent to the surface; for, if it meet in any other point, the plane must cut a second generator at this point, which is impossible, since no second generator meets an ideal-chord generator.

If, now, a section of the surface be made by an arbitrary plane through one of the four tangent generators, for example, through the generator r which is tangent to the double curve at the point R, the cubic branch of the curve of section will cut this generator in the point at which the plane is tangent to the surface, and will be tangent to this generator at the point R. If the plane touch the surface at R, this point becomes the double point of the cubic curve of section.

15. At every point in which the twisted cubic k^3 lies actually on the surface F_1^4, two real generators of the surface intersect. The curve of section of the plane of these generators consists of these two straight lines and a conic, since the plane cuts the two sheaves Φ_1 and Φ_1' in two projective sheaves of lines of the second order which have two self-corresponding rays, and which generate in addition to these rays a curve of the second order which is perspective to both sheaves of lines.* If the two generators through any point P meet the double curve a second time in the points Q and R, respectively, the conic of section must pass through these two points, since in each of them the plane of section cuts a second generator. The conic must also intersect each of the generators PQ and PR in a second point, at which points the plane of section is tangent

* Reye, G. d. L., I, S. 137.

to the surface, so that the plane of two generators is a bitangent plane to the surface. Every bitangent plane must contain two generators, and therefore cuts the surface along these generators and in a conic.

If one of the generators, PQ, be a tangent generator, the conic will pass through the point P, while the point of tangency along the generator PR will coincide with P, but the point of tangency along the tangent generator will in general be different from P. Suppose one of the generators through P pass through a pinch point A, then must the conic be tangent to the generator PA at A. If, finally, the bitangent plane contain the two consecutive generators through a pinch point P, these generators meeting the double curve a second time at Q, then it also contains the tangent to k^3 at Q, and the remaining part of the curve of section is a conic passing through Q. This plane is tangent to the surface all along the generator PQ except at the pinch point, the generator therefore being *torsal*. Through each point of the double curve k^3 there pass in general three distinct bitangent planes.

16. Let the two conics arising from the section of the surface F_1^4 by two arbitrary bitangent planes, σ and σ', be denoted by f_1 and f_1', respectively. These are related projectively to one another, since each is perspective to both sheaves of planes which generate F_1^4, and determine completely a collinear relation between the planes in which they lie, and in such a manner that these planes have no self-corresponding point. For each point of the conic f_1 corresponds to one and but one point of the conic f_1', namely, those points which lie in the same generator of F_1^4 are corresponding points. Whenever two generators of the surface intersect, the four points in the conics f_1 and f_1' arising from these must lie in one plane, the bitangent plane through the two generators. If, now, the planes σ and σ' have a self-corresponding point C, any ray of σ which passes through C and cuts the conic f_1 in two points, P and Q, corresponds to a ray of σ' which also passes through C, and which cuts the conic f_1' in two points P' and Q', corresponding to P and Q respectively. The generators PP' and QQ' consequently lie in a plane determined by a pair of corresponding rays through C, and similarly, each pair of generators which intersect lies in a plane which passes through C; that is, all the bitangent planes of the surface F_1^4 pass through one point, which is evidently impossible. Hence the planes σ and σ' have no self-corresponding point, and therefore generate a sheaf of planes of the

third order to which the bitangent planes of the surface F_1^4 belong, since each bitangent plane intersects σ and σ' in a pair of corresponding lines.*

17. The generators of the surface F_1^4 are not only the lines of intersection of corresponding planes in the sheaves of the second order Φ_1 and Φ_1', but are at the same time lines joining corresponding points in the projective conics f_1 and f_1', so that the surface may be generated as well by the projective conics as by the projective sheaves of planes. And since the collinear planes σ and σ', in which f_1 and f_1' are corresponding figures, have no self-corresponding element, the method of generating the surface by means of the projective conics is the exact reciprocal of the method by which the surface was originally generated; that is, by means of the projective sheaves of planes. Hence the surface F_1^4 is its own reciprocal.†

18. It has been already shown that the conic in which the surface F_1^4 is cut by an arbitrary bitangent plane intersects the double curve of the surface in two points, through which pass the two generators lying in the plane of the conic, one generator through each point, and two other generators which are cut in these points by the bitangent plane. Through every other point of this conic there passes one generator of the surface, and consequently no chord of the twisted cubic which is not a generator of the surface. From this it follows immediately that *those chords of the twisted cubic curve k^3 which intercept a conic having two points in common with the cubic, form the system of generators of the most general ruled surface of the fourth order.* Through each of the points of intersection of the conic with the cubic there passes an infinite number of chords which project the points of the cubic, and which consequently lie on cones of the second order. Only two of the rays of each of these cones, however, belong to the surface F_1^4.

19. That the chords of the twisted cubic curve k^3 which meet an arbitrary conic r^2 having two points, A and B, but no third point in common with the

* Reye, G. d. L., II, S. 94.

† This proof that the surface F_1^4 is of the same form as its reciprocal proceeds upon the supposition that two real bitangent planes to the surface exist. But this is not the case in some of the subforms, as for example in the subform (17). However, if we study the reciprocal method of generating the surface, it is readily seen that the generating conics may be so chosen in the collinear planes that these same subforms arise.

cubic form the system of generators of a ruled surface of the fourth order, may be shown directly in the following manner:

Out of any point S of k^3, not lying in the plane of r^2, project the points of this conic by means of the rays of a cone of the second order, P_1, and find in the bundle S', whose centre is likewise a point of k^3 not lying in the plane of r^2, the corresponding cone P'_1. Two pairs of corresponding rays of these cones will intersect in the points A and B, namely, SA and $S'A$ are corresponding rays, SB and $S'B$ are corresponding rays. Also, out of S' project the points of r^2 by means of the rays of a cone of the second order, P'_2, and find in the bundle S the corresponding cone P_2. The rays SA and $S'A$, SB and $S'B$ belong as well to these cones, P_2 and P'_2, as to the cones P_1 and P'_1. Thus we have in the bundle S two cones P_1 and P_2 which have two common rays SA and SB, and in the bundle S' the two cones P'_1 and P'_2 which correspond to these, and which also have two common rays, $S'A$ and $S'B$.

Now the two cones P_1 and P'_2 are both perspective to the conic r^2, and consequently we may establish a projective relation between them by correlating to one another those rays which pass through the same points of r^2. But the rays of the cones P_1 and P'_1, as also the rays of P_2 and P'_2, are related projectively to one another by virtue of their correspondence in the two bundles. By this means is established a projective relation between the rays of the cones P_1 and P_2, as also between the rays of P'_1 and P'_2, and in such a manner that the rays SA and SB are self-corresponding rays of the cones P_1 and P_2, and $S'A$ and $S'B$ are self-corresponding rays of the cones P'_1 and P'_2.

The plane of the bundle S which is determined by two corresponding rays of the cones P_1 and P_2 corresponds to the plane of the bundle S' which is determined by the rays corresponding to these, and which also correspond to one another, in the cones P'_1 and P'_2. Therefore, these two planes intersect in a chord of k^3, and this chord must meet the conic r^2, since the two planes pass through the same point of r^2. For example, let E be any point of the conic r^2, e_1 the ray of P_1 which passes through E, and e'_1 the corresponding ray of P'_1; also, let e'_2 be the ray of P'_2 which passes through E, and e_2 the corresponding ray of P_2. Then will e_1 and e_2 be corresponding rays of the cones P_1 and P_2, e'_1 and e'_2 will be the rays corresponding to these, respectively, and to one another in the cones P'_1 and P'_2, and the plane of the bundle S determined by the rays e_1 and e_2 will correspond to the plane of the bundle S' determined by the rays

e_1' and e_2'. These planes will intersect in a chord of k^3 which passes through E, for in each plane there lies a ray which passes through that point.

Since E is any point of the conic r^2, through every point of this conic there passes a chord of k^3 which is projected out of S by a plane containing a pair of corresponding rays of the cones P_1 and P_2, and out of S' by a plane which contains the rays of P_1' and P_2' corresponding to these, and to one another. No chord of the cubic can meet the conic r^2 which is not so projected out of S and S', since in no point of the conic can two chords intersect, the points A and B being exceptional points, through each of which there passes an infinite number of chords which lie, in each case, on a cone of the second order projecting the cubic.

Let π be an arbitrary plane which cuts the cones P_1 and P_2, P_1' and P_2' in the conics p_1^2 and p_2^2, $p_1'^2$ and $p_2'^2$, respectively. Of these conics, p_1^2 and p_2^2 are related projectively to one another and have two points of intersection as self-corresponding points, namely, those points in which the two self-corresponding rays SA and SB of the cones are cut by the plane π. The system of straight lines joining pairs of corresponding points in these two conics forms a sheaf of lines of the second order perspective to both conics,* and hence the planes of the bundle S determined by pairs of corresponding rays of the cones P_1 and P_2 form a sheaf of planes of the second order, Φ_{12}, perspective to both cones. Similarly, the conics $p_1'^2$ and $p_2'^2$ generate a sheaf of lines of the second, and the cones P_1' and P_2', of which these conics are sections, generate in the bundle S' a sheaf of planes of the second order, Φ_{12}', perspective to both cones P_1' and P_2', and therefore projectively related to the sheaf of planes Φ_{12}. Corresponding planes in these two sheaves pass through the same point of r^2, and conversely, through every point of r^2 goes a pair of corresponding planes of these sheaves. Hence the sheaves of planes Φ_{12} and Φ_{12}' generate a ruled surface of the fourth order, each of whose generators is a chord of the cubic k^3 which meets the conic r^2, while through every point of r^2 there passes a generator of this surface.

The conic r^2 must lie entirely outside the cones K_{12} and K_{12}', which are enveloped by the sheaves of planes Φ_{12} and Φ_{12}', respectively, otherwise it would not be possible for a generator of the surface arising from these sheaves to pass through every point of r^2. Two planes of each sheaf pass through each point of r^2, only one pair of which are corresponding planes, however, except in the points

* Reye, G. d. L., I, S. 137.

A and B, where both pairs are corresponding. The twisted cubic k^3 must lie actually upon this surface, at least in the points A and B and in the third point of intersection of the plane of the conic r^2 with the cubic, since through this third point two chords of the cubic pass which intercept the conic in A and B, respectively, and in two other points.

§3.—*The Surface with a Double Twisted Cubic and a Single Straight Line Director.*

20. It was shown in Art. 6 that all the chords of the twisted cubic k^3 which meet an arbitrary straight line r having no point in common with the cubic, are projected out of any two points of this curve, S and S', by means of two projective sheaves of planes of the second order; consequently these chords of the cubic form the system of generators of a ruled surface of the fourth order. Let this surface be denoted by F_2^4. Since an arbitrary plane through the straight line r cuts the curve k^3 in general in three distinct points, and contains three chords of this curve, the generators of the surface F_2^4 in general lie by threes in planes through the straight line r. The twisted cubic k^3 is, as in the most general case, a double curve on the surface.

On the other hand, if any plane section of the ruled surface generated by two projective sheaves of planes of the second order, chosen arbitrarily in the collinear bundles S and S', contain three generators of this surface, then must every generator of the surface meet some one straight line lying in this plane. For this plane of section cuts the sheaves generating the surface in two projective sheaves of lines of the second order, which have three self-corresponding rays. The straight line r joining the points of intersection of any two pairs of corresponding rays in these sheaves of lines also meets the three self-corresponding rays, and therefore contains five, and consequently all points of intersection of pairs of corresponding rays. Hence all generators of the surface will meet this one straight line r.

Since common rays of two sheaves of lines of the second order lying in the same plane occur in pairs, and the two sheaves of the preceding paragraph have three self-corresponding rays, there must be some fourth ray, d, which is common to both. This common ray, d, is cut by the remaining rays of the two sheaves in two projective ranges of points which, however, have three self-corresponding

points, and which consequently coincide throughout. All pairs of corresponding rays in the two sheaves therefore intersect in the points of the straight line d; but they also intersect in the points of r, which is impossible unless d and r coincide. The common ray d cannot be a self-corresponding ray of the two sheaves, for then would four chords of the twisted cubic lie in one plane. The straight line r, therefore, which is met by every generator of the surface, must be the line of intersection of two non-corresponding planes of the generating sheaves, and is consequently a common ray of those two sheaves of lines in which the generating sheaves of planes are cut by any plane through r. Through every point of r there passes one and only one generator of the surface, since in every point of d, one and only one pair of corresponding rays of the sheaves of lines intersect. The line r is therefore a single director line on the surface, through every point of which there passes a generator. In no case can this line meet the twisted cubic, for if so, through the point of intersection would pass more than one generator of the surface.

Thus it has been shown that *those chords of a twisted cubic curve which meet an arbitrary straight line having no point in common with this curve, form the system of generators of a Ruled Surface of the Fourth Order on which the twisted cubic lies as a double curve, and which is such that an arbitrary plane passed through the straight line director contains in general three generators of the surface; and conversely, if any three generators of a Ruled Surface of the Fourth Order which has a twisted cubic for its double curve lie in one plane, then every generator of that surface will meet one straight line which has no point in common with the cubic.*

21. The sheaves of planes Φ_2 and Φ_2' which generate the surface F_2^4 occupy a wholly special position with respect to the cubic curve k^3. Each of the two generators, AB and AC, which pass through any point A of the double cubic, and which meet this curve a second time in the points B and C, respectively, is tangent to both cones K_2 and K_2' which are enveloped by the sheaves Φ_2 and Φ_2'. In order that the chord BC, the only remaining chord of the cubic to be found in the plane of AB and AC, may also be a generator of the surface F_2^4, in which case, as was shown above, every generator of the surface will meet one straight line in this plane, the sheaves Φ_2 and Φ_2' must be so chosen that this chord, as well as the chords AB and AC, is tangent to the cones K_2 and K_2'; that is, this chord must also lie in a pair of corresponding planes of the sheaves

Φ_2 and Φ_2'. The sheaf of planes Φ_2 is therefore so situated with respect to the cone K which projects the cubic curve out of S, and similarly, the sheaf Φ_2' to the cone K', that three planes of the sheaf intersect, two and two, in three rays of the cone. It is well known that when one such triplet of planes exists in the sheaf Φ_2 there is necessarily an infinite number of such triplets;* consequently, if there be on the surface F_2^4 one set of three generators which lie in the same plane, there will necessarily be an infinite number of such sets, and the planes in which these sets lie must all pass through that line which is met by every generator of the surface.

If there be any ideal-chord generators on the surface F_2^4, these will lie each in one plane through the straight line director. The planes through this director will thus be separated into groups, each of the planes in one group containing three actual-chord generators, those in the next consecutive group containing each one ideal-chord generator, there being at most two groups of each kind, and the groups being separated by planes which contain in general one tangent generator and one actual-chord generator, it being possible, however, for these to coincide as a single tangent generator.

It is readily seen that when two of the generators lying in one plane through the straight line director approach indefinitely near to each other, as in the case of the boundary plane between two groups, these consecutive generators pass through one of the pinch points on the surface, and that the third generator of this plane is tangent to the double curve. In the first place, when two consecutive generators intersect in a point of the double curve this point is always a pinch point, since the two tangent planes to the surface at this point are coincident. But aside from this, if two of the three generators lying in one plane through the straight line director approach indefinitely near to each other, two of the three planes of the sheaf Φ_2 which give rise to this set of generators must also approach indefinitely near to each other, in which case the third plane of this triplet becomes tangent to the cone K. But this can only occur when the two consecutive planes of this triplet pass through a ray common to the cones K and K_2. Similar relations hold for the sheaf Φ_2' and the cones K' and K_2'. Hence the two consecutive generators pass through a pinch point while the third generator of the set to which these belong is a tangent generator.

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* Compare Cremona, Elem. of Proj. Geom., p. 244; Poncelet, Proj. Prop., art. 565.

22. Any plane through the straight line director of this surface F_2^4 is evidently tangent to the surface at every point in which a generator lying in this plane intersects the director, and hence each plane through the straight line director is in general a tritangent plane to the surface. The points of tangency of a tritangent plane vary along the director as the plane rotates about this line. They remain distinct, or two, or all three of them coincide, or two of them become imaginary, according as the generators lying in the plane remain distinct, or two, or all three of them coincide, or two of them become imaginary. Every bitangent plane of the surface F_2^4 is thus a tritangent plane, while the three bitangent planes which in the general surface F_1^4 pass through any one point of the double curve coincide in a single plane through r. The bitangent planes of this surface therefore form a triply counting sheaf of the first order whose axis is the straight line director.

23. Unlike the general surface F_1^4, this species F_2^4 differs in form from its reciprocal. The reciprocal surface which may be denoted by $F_{2'}^4$ must satisfy the conditions that through each point in which two generators intersect a third shall pass; these triple points all lie in one straight line; any plane through this straight line contains but one generator of the surface; the planes containing two generators of the surface, i. e. the bitangent planes, form a sheaf of the third order. This surface $F_{2'}^4$ is thus a triple line surface, and cannot be generated by means of two projective sheaves of planes of the second order.

24. A ruled surface of the fourth order with a non-degenerate twisted cubic for its double curve cannot have more than one straight line director. For, if two, then any plane through one of them contains in general three generators of the surface which meet the second director and which do not pass through one point. Hence the two directors must lie in one plane, and consequently all generators of the surface would lie in one plane, which is evidently impossible.

§4.—*The Surface with a Double Conic and a Double Straight Line which is not a Generator.*

25. Let us next consider the case in which the bundles S and S' have one self-corresponding plane, but no other self-corresponding element. The line congruence arising from these bundles is, as we have seen, of the first order and

second class, while the curve of singular points consists of a conic k^2 and a straight line k which meet in a point O, but which are not in the same plane.

As before, select at will in the bundles S and S' a pair of corresponding sheaves of planes of the second order, Φ_3 and Φ_3'. These are related projectively to each other and generate, as in the general case, a ruled surface of the fourth order, F_3^4. Each generator of the surface is a line of the congruence, and is tangent to both the cones K_3 and K_3' which are enveloped by the sheaves Φ_3 and Φ_3', respectively. But since any line of the congruence which is tangent to the one cone K_3, the lines through the vertex being generally excluded from our consideration, is necessarily tangent to the other cone K_3', and all such lines of the congruence lie in pairs of corresponding planes of the sheaves Φ_3 and Φ_3', the surface F_3^4 is completely defined when we say that *those lines of a congruence of the first order and second class which are tangent to a cone of the second order whose vertex is a point of the singular conic of the congruence form the system of generators of a ruled surface of the fourth order.*

Every straight line which meets the conic k^2 and the straight line k is a ray of the congruence, and consequently the surface F_3^4 may be considered as generated by a straight line which constantly meets both k^2 and k, and which moves so as to continually touch a cone of the second order whose vertex is a point of the conic k^2.

26. Through each point of k^2 and of k in general pass two pairs of corresponding planes of the sheaves Φ_3 and Φ_3', and consequently two generators of the surface F_3^4. Hence the conic k^2 and the straight line k are double curves on the surface, the two generators through any point of these curves being real and distinct, coincident or imaginary according as the point lies without, on or within the cones K_3 and K_3'. These cones in general intercept the double conic k^2 and the double straight line k in four points, two of which are on the conic and two on the straight line. These four points of intersection are, as in the most general case, the pinch points of the surface.

27. The two generators through any point of the nodal conic lie in a plane with the nodal straight line. The generators through O, the point of intersection of k^2 and k, meet neither branch of the double curve elsewhere, and lie in the plane determined by k and the tangent to k^2 at this point O. The point O is

thus a "singular" point on the double curve. The pinch points on each branch of the double curve divide the branch into two segments, of which the one lies actually upon the surface, while the other segment is isolated. If the singular point O appears in that segment of the nodal conic which lies actually upon the surface, it must likewise appear in the segment of the nodal straight line which lies actually upon the surface, since in this case it lies outside the cones K_3 and K'_3, and similarly, if the singular point appears in the isolated segment of the one branch, then also in the isolated segment of the other branch. If a pinch point on one branch of the double curve coincide with the singular point O, so also must one of the pinch points on the other branch coincide with this singular point, and beside this point there will always exist one real pinch point on each branch of the double curve.

28. There are then five subforms of the surface F_3^4 to be considered, as follows:

(1). The pinch points are all real and distinct. The cones K_3 and K'_3 intercept k^2 and k, each in two distinct points. Each branch of the double curve has one segment which lies actually upon the surface, and one isolated segment. This subform may be further subdivided according as the singular point O lies actually upon the surface, or appears in the isolated segments of the double curve.

(2). The pinch points on the nodal conic are real and distinct, but on the nodal straight line are imaginary. The cones K_3 and K'_3 intercept the conic k^2 in two points, but do not intercept the straight line k. The nodal line lies actually upon the surface throughout, the conic has one segment which lies actually upon the surface, and one isolated segment. The singular point O must appear in that segment of the conic which lies actually upon the surface.

(3). The pinch points on the nodal conic are imaginary, but on the nodal straight line are real and distinct. The cones K_3 and K'_3 intercept the straight line k in two points, but do not intercept the conic k^2. The nodal conic lies actually upon the surface throughout, while of the nodal straight line there is one isolated segment and one segment which lies actually upon the surface, the singular point O always appearing in this latter segment.

(4). One pinch point on each branch of the double curve coincides with the singular point O; a second pinch point exists on each branch. The cones K_3 and K'_3 pass through the singular point O, and consequently intercept both k^2 and k in a second point. There is one isolated segment on each branch of the double curve, the singular point O forming one of the boundaries of each segment.

(5). The pinch points are all imaginary. The cones K_3 and K'_3 are so situated as not to intercept either k^2 or k. Both nodal conic and nodal straight line lie actually upon the surface at every point.

29. The section of the surface F_3^4 by an arbitrary plane is, as for the most general surface F_1^4, a quartic curve which has in general three double points. The curve of section by a plane through a single arbitrary generator consists of this generator and a cubic with a double point, and since every generator of this surface meets both the nodal conic and the nodal straight line in real points, the cubic branch of the curve of section must intersect the generator in three real points, namely, in the points P and Q in which the generator meets k^2 and k, respectively, and in a third point R at which the plane is tangent to the surface. When the point of tangency falls in with the point P this point becomes the double point on the cubic, and the plane contains the tangent to the conic at this point. The tangent plane to the surface at any point of the nodal conic is thus determined by the tangent to the conic at that point and a generator passing through the point. At each point of the nodal conic there are in general two distinct tangent planes to the surface. Only at the pinch points and at the singular point O do these two planes coincide, while for points in the isolated segment the tangent planes are imaginary.

When the point of tangency of the plane through a generator PQ coincides with the point Q in which the generator intersects the nodal line k, the plane must contain this nodal line, since it contains three consecutive points of the line. The curve of section in this case consists of the generator PQ, the second generator PQ' through the point P of the conic, and the straight line k counting doubly, since in each point of k two generators in general are intercepted. Every plane through the nodal line k contains in general two generators which intersect in a point of the nodal conic, and is therefore tangent to the surface at two points on k, namely, the points of intersection of these generators with k,

these being in general distinct. Moreover, at every point of the nodal line k there are two planes which are tangent to the surface at this point; these may, however, coincide or be imaginary.

30. The plane of the two generators through any point of the nodal conic is thus a bitangent plane to the surface passing through the nodal line k, the totality of such planes forming a sheaf of the first order whose axis is k. Since the two generators passing through a pinch point on the nodal conic coincide, the points of tangency of the plane determined by this doubly counting generator and the nodal line k also coincide and the plane is tangent to the surface all along this generator, which is consequently torsal. The tangent plane at the pinch point, however, is as usual the plane of the generator and the tangent to the conic at this point.

The plane of the two generators through any point Q of the nodal line k cuts the nodal conic in two points P and P' in which the two generators meet the conic. This plane cuts the sheaves Φ_3 and Φ_3' generating the surface in two projective sheaves of lines of the second order which have two self-corresponding rays, and which consequently generate a curve of the second order perspective to both sheaves of planes. This conic is met by every generator of the surface and passes through the two points P and P' of the nodal conic, since in these points generators not lying in the plane of section are intercepted by this plane. The conic of section also intersects the two generators of its plane in P and P', and in two additional points at which the plane cuts the generators consecutive with these, that is, at which the plane is tangent to the surface. Every plane therefore through two generators which intersect in a point of the nodal line k is a bitangent plane, and cuts the surface along these two generators and in a conic which passes through the points of tangency of the plane, and which meets the nodal conic in two points.

The plane of the two coincident generators at a pinch point on the nodal line is tangent to the surface all along this generator except at the pinch point, and contains the tangent to the nodal conic at the point P in which this torsal generator meets the nodal conic. The curve of section of this plane consists of the torsal generator counting doubly, and a conic which meets the nodal conic k^2 in two consecutive points, and which consequently has a common tangent with k^2 at this point. The plane of the two generators through the singular

point O cuts the surface along these two generators and along the nodal line k, which as a part of the curve of section counts doubly. The two points of tangency of this plane coincide at the singular point; the two tangent planes at the singular point likewise coincide in this plane. The singular point O is thus in a certain sense a pinch point, but differs from an ordinary pinch point in that the two generators through it do not in general coincide. The curve of section of the plane of the nodal conic consists simply of this conic counting doubly.

31. The bitangent planes to this surface which are determined by pairs of generators intersecting in a point of k form a sheaf of planes of the second order. Let α and α' be any two of these bitangent planes which meet k in the points Q and Q', and contain the pairs of generators a, b and a', b', respectively. Denote the conics of section lying in these planes by a^2 and a'^2, and let the generators a and b meet the conic a^2 of their plane in the points A and B, respectively, at which points the plane α is tangent to the surface; similarly, let the generators a' and b' meet the conic a'^2 of their plane in A_1' and B_1', respectively, at which points α' is tangent to the surface. Now, the generators a and b must meet the conic a'^2 in real points, but this can only occur in the points in which a'^2 cuts the line of intersection, l, of the planes α and α'. Denote these points of a'^2 by A' and B'.

Since the conics a^2 and a'^2 are both perspective to the sheaves Φ_3 and Φ_3', they are projectively related to each other, those being corresponding points which lie on the same generator of the surface. Consequently A and A', B and B' are pairs of corresponding points of these projective conics. The conics determine a collinear relation between the planes α and α' such that the line AB of α corresponds to the line $A'B'$, or l, of α', and these lines are distinct, so that of the planes α and α' there cannot be more than one self-corresponding point. An arbitrary plane through k contains in general two generators of the surface F_3^4, and consequently two pairs of corresponding points of the conics a^2 and a'^2, and therefore this plane cuts α and α' in a pair of corresponding lines, and the lines AB and l in a pair of corresponding points. The point L therefore in which AB and l intersect is a self-corresponding point of the planes α and α'.

Through this self-corresponding point must pass each bitangent plane which contains two generators intersecting in a point of k; for, every such plane cuts α and α' in a pair of corresponding lines which intersect in a point of l, but this

is only possible when the plane passes through the self-corresponding point L. The totality of such bitangent planes therefore cuts α and α' in two projective sheaves of lines whose centres coincide at L and which have no self-corresponding ray. Hence these bitangent planes form a sheaf of the second order whose centre is the point L.

Thus the bitangent planes of the surface F_3^4 lie in two distinct sheaves, one of the first order whose axis is k, and the other of the second order.

32. Since every generator of the surface F_3^4 meets the conic of section lying in any bitangent plane, and through every point of this conic there passes a generator, the surface F_3^4 may be defined as being formed from *those rays of a congruence of the first order and second class which meet a conic having two points in common with the singular conic* k^2, *the two points being distinct from the singular point* O; or it may be defined as the surface generated by a straight line moving so as always to meet two conics k^2 and m^2, which lie in different planes and have two points in common, and a straight line k lying in a plane with neither conic but having a point in common with one of them. It is readily shown, conversely, as in the most general case, that those rays of the congruence which intersect a conic having two points in common with k^2 are projected out of S and S' by means of two projective sheaves of planes of the second order, and consequently that these rays form the system of generators of a ruled surface of the fourth order.

33. It has already been noticed that the two conics of section lying in two arbitrary bitangent planes which contain pairs of generators intersecting in points of k are projectively related to each other, and determine by their projectivity the collinearity of the planes in which they lie in such a manner that these planes have one and but one self-corresponding point. The generators of the surface F_3^4, therefore, may be considered as the lines joining pairs of corresponding points in two corresponding conics chosen in collinear plane fields which have one self-corresponding point. But this method of generating the surface F_3^4 is the exact reciprocal of that by which the surface was originally generated. Hence the surface which is reciprocal to F_3^4 is of the same form as F_3^4, or the surface F_3^4 is its own reciprocal.

34. The sheaves Φ_3 and Φ'_3 must be so chosen as not to include the self-corresponding plane α of the bundles S and S'. For any line of this plane which passes through the singular point O might then be considered a generator of the surface F_3^4, and the plane would contain an infinite number of generators and would therefore belong wholly to the surface. An arbitrary straight line would always intercept a generator lying in the plane α, and beside this would meet in general three and only three generators of the surface. Hence, if the sheaves Φ_3 and Φ'_3 include the self-corresponding plane α, the surface F_3^4 breaks up into this plane and a ruled surface of the third order.

Moreover, if the plane α belong to the sheaves Φ_3 and Φ'_3 the singular line k will be a double line on the surface as in the general case, but through each point of the singular conic there will pass only one generator aside from the generators lying in the plane of the conic, that is, only one generator of the cubic surface. The plane γ of any two generators which intersect in a point of the nodal line k will cut the sheaves Φ_3 and Φ'_3 in two projective sheaves of lines of the second order which have three self-corresponding rays, namely, the two generators lying in the plane and that line in which γ cuts the plane α. Hence every generator of the surface arising from these sheaves meets some one straight line g lying in the plane γ, which does not meet either k^2 or k. Conversely, if every generator of the surface F_3^4 meet some one straight line g different from the nodal line k, then must the sheaves of planes generating this surface include the self-corresponding plane α. For, through every point of this line there passes a generator, in particular through the point P where g meets the plane α. The ray projecting this point out of S is a ray of α and corresponds to some ray of S' which also lies in α. Similarly, the ray projecting the point P out of S' corresponds to some ray of S lying in α. Hence the generator through P must be the line of intersection of the plane α with itself; that is, the plane α is a self-corresponding plane of the two sheaves generating the surface. Therefore that variety of the surface F_3^4 which has a single straight line director in addition to the nodal conic k^2 and the nodal straight line k degenerates into a ruled surface of the third order and a plane, namely, the plane of the nodal conic.

It will be observed that this latter is the case which arises when the two pinch points on the nodal conic of the surface F_3^4 coincide, for then must the cones K_3 and K'_3 be tangent to the conic k^2, and hence the plane of the conic becomes a plane of the sheaves Φ_3 and Φ'_3 generating the surface.

§5.—*The Surface with a Double Conic and a Generator which intersects every other Generator.*

35. If now the generating sheaves of the last section be so chosen as to avoid the self-corresponding plane α, but so that one of planes of Φ_3 contains the singular line k, then the corresponding plane of Φ_3' will also pass through this line, and hence k will be a generator of the surface. This gives rise to what may be considered a distinct variety of the surface which we shall designate by F_4^4, and the sheaves of planes generating it by Φ_4 and Φ_4', respectively; that is, the ruled surface F_4^4 is of the fourth order, and has a nodal conic k^2 and a nodal straight line k which is also a generator of the surface.

Through each point of the nodal conic there pass in general two distinct generators of the surface which meet the nodal line k. One of these two generators which pass through the singular point O, however, coincides with the line k, while the other remains distinct, and lies in that plane through k which is tangent to k^2. Through each point of the nodal line k there pass two generators of this surface, the line k itself and a second generator which meets the nodal conic. In one single point of this line there is but one generator, the line k itself. In other words, at one single point of the line k the second generator coincides with k; this point is therefore a pinch point of the surface. Thus every plane through k contains in general three generators of the surface, namely, the line k and the two generators through that second point of the nodal conic which lies in the plane

36. The curve of section of the surface F_4^4 by an arbitrary plane through the nodal line k consists of the three generators lying in the plane, the generator k counting doubly, since in the points of this line the remaining generators are cut by the plane of section. The section of the surface by that plane through k which is tangent to the nodal conic consists of the line k counting triply and the second generator through O. This plane is tangent to the surface all along the line k, this line being a torsal generator. At every point along the line k, except at the pinch point and at the singular point O, there is a second tangent plane, namely, the plane of k and the second generator through the point. But every such plane contains also a third generator, and is consequently a tritangent plane to the surface, the points of tangency being the pinch point on k and those two points of k in which the remaining two generators of the plane inter-

sect this line. Every bitangent plane of this surface is therefore a tritangent, the points of tangency all lying on k, and the totality of such planes forming a sheaf of the first order whose axis is k.

37. This variety of the surface arises when the two pinch points of the more general surface F_3^4 which lie in the nodal line k coincide, for then is this line tangent to the cones K_3 and K_3', and hence lies in a pair of corresponding planes of Φ_3 and Φ_3'. The surface F_4^4 has then always one real pinch point on the nodal line k, as we have already seen, and may have either two or none on the nodal conic. A special case arises when one of the pinch points on the nodal conic coincides with the singular point O, in which case the pinch point on the nodal line also coincides with this point.

38. This surface, like the surface F_2^4, differs in form from its reciprocal. The reciprocal surface, which may be denoted by $F_{4'}^4$, must be such that through each point of intersection of two generators a third will pass, and always the same third generator; that is, the surface $F_{4'}^4$ is a triple line surface, the triple line being a generator through each point of which two other distinct generators pass. Each plane through the triple line contains but one other generator.

§6.—*The Surface with two Double Straight Line Directors and a Double Generator.*

39. Let us next consider the case in which the ray SS' of the bundles S and S' is a self-corresponding ray. As has been already observed (Art. 7), these bundles generate a congruence of the first order and first class, whose singular points lie on two straight lines k_1 and k_2, which are gauche to one another and meet the ray SS' in the points O_1 and O_2, respectively, every line of the congruence meeting both these singular lines, and conversely, every line which meets both k_1 and k_2 being a line of the congruence. The singular lines or axes k_1 and k_2 are in general distinct, but may coincide or become imaginary. We shall first consider the case in which the axes are real and distinct.

40. In S and S' select at will a pair of corresponding sheaves of the second order Φ_5 and Φ_5', and denote the corresponding cones enveloped by these sheaves by K_5 and K_5'. These sheaves of planes generate a ruled surface of the fourth order, F_5^4, whose generators are lines of the congruence.

Through each point of k_1 and k_2 in general pass two pairs of planes of Φ_5 and Φ_5', and hence two generators of the surface F_5^4, so that these lines are double or nodal lines on the surface. The two generators through any point of either nodal line are as usual real and distinct, coincident or imaginary according as the point lies without, on, or within the cones K_5 and K_5', it having been shown in the most general case, and is equally true in this, that any singular point which lies without, on, or within one of these cones has a similar position with respect to the other. The cones K_5 and K_5' in general intercept each of the singular lines in two points, thus dividing these lines each into two segments, of which the one lies actually on the surface at every point, while the other is isolated. The four points in which the cones meet k_1 and k_2 are as usual pinch points on the surface. Thus the surface F_5^4 has in general four pinch points, of which two lie on each nodal line.

If the point O_1 in which SS' meets the nodal line k_1 lies outside the cones K_5 and K_5', then also does the point O_2 in which SS' meets the line k_2, and if O_1 lie either on or inside these cones, so also does O_2. Any plane of Φ_5 which intercepts k_1 in the point O_1 contains the line SS', and consequently intercepts k_2 in O_2. Its corresponding plane in Φ_5' also passes through the line SS', so that SS' is a generator. In general, two pairs of corresponding planes in Φ_5 and Φ_5' pass through O_1 and O_2, hence SS' is in general a double generator of the surface F_5^4. This double generator is real or imaginary according as O_1 and O_2 lie outside or inside the cones K_5 and K_5'. If the pinch points on either nodal line k_1 or k_2 be imaginary, that is if the sheaves Φ_5 and Φ_5' be so chosen that the cones K_5 and K_5' do not intercept one of these lines, two pairs of corresponding planes of the sheaves must always pass through O_1 and O_2, and consequently the double generator must in this case be real.

41. The surface F_5^4 admits then the following subforms:

(1). The pinch points are all real, the double generator is real. In this case the cones K_5 and K_5' intercept both k_1 and k_2, so that on each nodal line there is one segment which lies actually on the surface and one isolated segment, the points O_1 and O_2 appearing in those segments of the nodal lines which lie actually on the surface.

(2). The pinch points are all real, the double generator is imaginary. This subform differs from the preceding one only in that the points O_1 and O_2 appear in the isolated segments of k_1 and k_2, respectively.

(3). The pinch points are all real, while the double generator becomes torsal. Here the cones K_5 and K_5' cut the nodal lines in the points O_1 and O_2, respectively, and each of them in one other point. Thus two of the pinch points lie on the double generator, the remaining two being also real.

(4). The two pinch points on one nodal line are real, on the other imaginary, the double generator being necessarily real. In this case the cones intercept one nodal line in two points which must be different from O_1 and O_2, but do not intercept the other line. The nodal line on which the two real pinch points lie has one segment which lies actually on the surface and one isolated segment, the point O_1 or O_2, as the case may be, which lies on this line always appearing in the former segment. The other nodal line lies actually on the surface throughout.

(5). All four pinch points are imaginary, the double generator being necessarily real. The sheaves Φ_5 and Φ_5' are so chosen that the cones K_5 and K_5' cut neither nodal line. Both these lines lie actually on the surface throughout, and the double generator is real.

42. If the sheaves Φ_5 and Φ_5' which generate this surface be so chosen as to include either α_1 or α_2 as self-corresponding plane, in other words, be so chosen that the cones K_5 and K_5' are tangent to either of the singular lines k_1 or k_2, and hence so that the two pinch points on this line coincide, then will the surface F_5^4 degenerate into this self-corresponding plane and a ruled surface of the third order. For suppose, for example, that the sheaves include the plane α_1; then any line in this plane which passes through O_2 may be considered a generator of the surface, and hence the plane belongs wholly to the surface. An arbitrary straight line will always meet one generator of the surface in this plane, and can therefore meet at most three other generators without meeting every generator of the surface.

Under these circumstances the line k_1 is cut by the planes of the sheaves Φ_5 and Φ_5' in two coincident ranges of points which are perspective to these sheaves. Through each point of k_1 passes one and only one generator of the surface of the third order, since through each point of this line there passes one and but one pair of corresponding planes of the sheaves, aside from the self-corresponding plane α_1, while in general through each point of k_2 will pass two generators of this surface.

An arbitrary plane ρ through one of the generators which intersect the line k_2 in any point R cuts the sheaves Φ_5 and Φ_5' in projective sheaves of lines of the second order which have two self-corresponding rays, namely, the generator lying in the plane and that line in which the plane ρ cuts α_1. Hence these sheaves generate a conic r^2 which is met by every generator of the surface, and which consequently passes through the point R. This conic is perspective to both sheaves Φ_5 and Φ_5', and is therefore projectively related to the line k_1. Hence the surface of the third order arising from the sheaves Φ_5 and Φ_5' may be considered as having been generated by the conic r^2 and the straight line k_1, which are related projectively to each other, and lie in different planes, the line k_2 meeting the conic and appearing as a double director on the surface.

On the other hand, if the sheaves Φ_5 and Φ_5' be so chosen that both α_1 and α_2 are included as self-corresponding planes, then the cones K_5 and K_5' are tangent to both k_1 and k_2, so that the pinch points on each line coincide, and the surface generated by these sheaves breaks up into these two self-corresponding planes and a ruled quadric. The lines k_1 and k_2 are in this case both perspective to the sheaves Φ_5 and Φ_5', and are therefore projectively related to each other. These generate the ruled quadric, which is such that the line SS' cannot be a generator, since the point O_1 of k_1 corresponds to the point of contact of the cones K_5 and K_5' with k_2, and similarly, the point O_2 of k_2 corresponds to the point of contact of these cones with k_1. Moreover, it is evident that the surface F_5^4 can have no straight line director other than the two nodal lines without degenerating.

43. An arbitrary plane through a generator PQ of the non-degenerate surface F_5^4 cuts the surface along this generator and in a curve of the third order which has a double point at the intersection of the plane with the double generator. This cubic curve meets the generator lying in its plane in three real points, namely, in the points of intersection, P and Q, of the plane and generator with the nodal lines k_1 and k_2, respectively, and in a third point at which the plane cuts the consecutive generators and is tangent to the surface. If this point of tangency move along the generator till it coincide with the point P, the plane at the same time rotates about the generator till it contains the nodal line k_1. In this position the plane also contains the second generator through Q, and is tangent to the surface at the points of intersection of these two generators with

the nodal line k_1. The curve of section of this plane consists of the two generators lying in it and the nodal line k_1 counting doubly, since in each point of this line two generators are intercepted by the plane.

44. Every plane through a nodal line is a bitangent plane to the surface, since it contains two generators which intersect in a point of the other nodal line, and conversely, the plane of the two generators which meet in any point of one nodal line contains the other nodal line, and is tangent to the surface at the two points along this second line in which the two generators intersect it. The two points of tangency of the plane determined by either nodal line and the two consecutive generators which intersect in a pinch point on the other nodal line coincide, while the plane is tangent to the surface all along these generators except at the pinch point. At this point the tangent plane is determined by this torsal generator and the nodal line on which the pinch point lies.

On the other hand, at every point of either nodal line two tangent planes exist which are in general distinct, namely, the planes determined by the nodal line and each of the two generators passing through the point. Only at the pinch points and at the singular points O_1 and O_2 do these two tangent planes coincide. Although the two generators through the singular points O_1 and O_2 of this surface coincide, these points still differ from ordinary pinch points, since the two generators through them are not in general consecutive generators.

Since in general two pairs of corresponding planes of Φ_5 and Φ_5' intersect in the line SS', an arbitrary straight line which meets SS' can meet at most two other generators of the surface, hence an arbitrary plane γ through SS' cuts the surface F_5^4 in this doubly counting generator and in a conic g^2 which meets this generator in two points, and which is perspective to both sheaves Φ_5 and Φ_5'. The plane of the conic is tangent to the surface at the two points in which the conic and double generator SS' intersect, for at each of these points the plane intercepts generators which are consecutive to the double generator. Only when the arbitrary plane coincides with either of the singular planes α_1 or α_2 does the conic degenerate, in which case the curve of section consists of the double generator and the nodal line which lies in the plane, while the two points of tangency coincide at the intersection of these lines.

Thus the bitangent planes of the surface F_5^4 form three sheaves of the first order, whose axes are the nodal lines k_1 and k_2 and the double generator SS'.

45. Since each generator of the surface F_5^4 meets both straight lines k_1 and k_2 and the conic g^2, this surface may be generated by a straight line which moves so as always to meet two straight lines not lying in the same plane and a conic which neither meets nor lies in the same plane with either line. The double generator appears as the line joining the points in which the plane of the conic cuts the two arbitrary lines, and is real or imaginary according as the conic cuts this line in real or imaginary points.

46. Two arbitrary planes through the line SS' cut the surface F_5^4 in conics which are both perspective to the sheaves Φ_5 and Φ_5', and which are consequently projectively related to each other, and in such a manner that the points in which the one conic intersects the double generator correspond respectively to the points in which the other conic intercepts this line. Hence the surface F_5^4 may be generated by two projectively related conics which lie in different planes and have no point in common, but which are so situated that the points in which the one conic cuts the line of intersection of their planes correspond respectively to the points in which the other conic cuts this line. But this method of generating the surface is the reciprocal of that by which the surface was originally generated, the two projective conics determining the collinearity of the planes in which they lie in such a manner that their line of intersection is a self-corresponding line, but not every point of it a self-corresponding point. Hence the surface F_5^4 also is its own reciprocal.

§7.—*The Surface with One Double Straight Line Director and a Double Generator.*

47. Suppose now that the two self-corresponding planes in the bundles S and S' approach indefinitely near to each other and finally coincide in a single plane α_{12}. Then, as we have seen, the two singular lines k_1 and k_2 also coincide and lie in the plane α_{12} as one single line k_{12} containing all the singular points of the congruence outside the ray SS'. Through each point of k_{12} there passes an infinite number of lines of the congruence which lie in a plane with k_{12}.

Under these conditions a surface F_6^4 generated by a pair of corresponding sheaves of planes of the second order, Φ_6 and Φ_6', lying in the bundles S and S' is as before of the fourth order, and has the line k_{12} as a nodal line, while in general the line SS' joining the centres of the sheaves lies on the surface as a double generator. The cones K_6 and K_6' which are enveloped by the sheaves Φ_6 and Φ_6', respectively, intercept the line k_{12} in general in two points, both cones

passing through the same points of the line. These two points are the pinch points on the surface F_6^4, and may be either real or imaginary. The pinch points as usual divide the nodal line into two segments, of which the one lies actually on the surface at every point while the other is isolated. The double generator SS' is real or imaginary according as the point O_{12} in which SS' meets the nodal line k_{12} lies actually or ideally upon the surface.

If one of the two pinch points coincides with O, then is SS' a doubly counting torsal generator. When the two pinch points on the surface are imaginary, that is, when the cones K_6 and K_6' do not intercept the line k_{12} in real points, then must the line SS' be a real double generator. If, finally, the two pinch points coincide, which happens when the cones K_6 and K_6' are tangent to the line k_{12}, then is the plane α_{12} a self-corresponding plane of the generating sheaves, and hence the surface breaks up into this plane and a ruled surface of the third order.

48. Every plane through the nodal line k_{12} contains two generators which intersect in a point of this nodal line; that is, every plane through the nodal line is a bitangent plane to the surface, the two points of tangency coinciding in a point of the nodal line. Conversely, since the two generators which pass through any point of the nodal line lie in one plane through this line, the two planes which are tangent to the surface at this point coincide. Every point along the nodal line k_{12} is therefore a cuspidal point of the surface, but it is only at the pinch points and at the singular point O_{12} that the two generators through the point coincide. Here again the singular point O_{12} differs from an ordinary pinch point, as the generators through it are not in general consecutive generators. The curve of section of any plane through the nodal line consists of the two generators lying in that plane and of the nodal line counting doubly.

49. As in the surface F_5^4, any plane through the double generator cuts the surface along this generator and in a conic perspective to both the generating sheaves Φ_6 and Φ_6', and which meets the double generator in two points at which the plane is tangent to the surface. The bitangent planes of the surface F_6^4 thus form two ordinary sheaves of the first order whose axes are the nodal line k_{12} and the double generator SS'. It readily follows, as in the surface last considered, that the surface F_6^4 is likewise of the same form as its reciprocal.

50. Of the remaining four species of the Ruled Surface of the Fourth Order enumerated by Cremona, none can be generated by means of projective sheaves of planes of the second order, or by the reciprocal method of projective conics. Two of these, Cremona's ninth and tenth, Cayley's third and sixth, are self-reciprocal, triple-line surfaces, the ninth having a distinct single straight line director, the tenth without such a director. Both these surfaces admit trinodal quartic sections, the nodes, however, being coincident, forming a triple point. The former surface may be generated by means of a straight line and a plane cubic with a double point which are projectively related to each other, on condition that the point of the straight line which lies in the plane of the cubic corresponds to that third point of the cubic which lies in the plane determined by the generating line and the double point of the cubic; the latter surface, on condition that the generating line pass through the double point of the cubic.

The remaining two surfaces, Cremona's eleventh and twelfth, Cayley's first and fourth, do not admit a trinodal quartic section. They differ from the surfaces denoted by F_5^4 and F_6^4, respectively, in that they have no double generator, but have in general four pinch points on each nodal line.

CLARK UNIVERSITY, WORCESTER, MASS., *March 15th*, 1893.

Note on the so-called Quotient G/H in the Theory of Groups.

BY PROF. CAYLEY.

The notion (see Hölder, "Zur Reduction der algebraischen Gleichungen," Math. Ann., t. XXXIV (1887), §4, p. 31) is a very important one, and it is extensively made use of in Mr. Young's paper, "On the Determination of Groups whose Order is the Power of a Prime," Amer. Math. Jour., t. XV (1893), pp. 124–178; but it seems to me that the meaning is explained with hardly sufficient clearness, and that a more suitable algorithm might be adopted, viz. instead of $G_1 = G/\Gamma_1$ I would rather write $G = \Gamma_1 . QG_1$ or $QG_1 . \Gamma_1$.

We are concerned with a group G containing as part of itself a group Γ_1 such that each element of Γ_1 is commutative with each element of G. This being so, we may write

$$G = QG_1 . \Gamma_1,$$

where QG_1 is not a group but a mere array of elements, viz. if $\Gamma_1 = (1, A_2, \ldots . A_s)$, and $QG_1 = (1, B_2, \ldots . B_t)$, then the formula is

$$G = (1, B_2, \ldots . B_t)(1, A_2, \ldots . A_s),$$

where it is to be noticed that the elements B are not determinate; in fact, if A_θ be any element of Γ_1, we may, in place of an element B, write BA_θ, for $B(1, A_2, \ldots . A_s)$ and $BA_\theta(1, A_2, \ldots . A_s)$ are, in different orders, the same elements of G.

But, G being a group, the product of any two elements of G is an element of G; viz. we thus have in general

$$B_i A_{i'} . B_j A_{j'} = B_k A_{k'}\,; \text{ that is, } B_i B_j = B_k A_{k'} A_{j'}^{-1} A_i^{-1} \quad (i, j, \text{ unequal or equal}),$$

where the B_k is a determinate element of the series $1, B_2, \ldots . B_t$, depending only on the elements B_i and B_j into the product of which it enters; and it is in

nowise affected by the before-mentioned indeterminateness of the elements B: say B_i, B_j being any two elements of the series $1, B_2, \ldots . B_t$, we have the last preceding equation wherein B_k is a determinate element of the same series.

We may imagine a set of elements $1, B_2, \ldots . B_t$ for which, B_i, B_j being any two of them and B_k a third element determined as above, we have always $B_iB_j = B_k$, that is these elements $1, B_2, \ldots . B_t$ now form a group, say the group G_1; the original elements $1, B_2, \ldots . B_t$, (which are subject to a different law of combination $B_iB_j = B_kA_{k'}A_{j'}^{-1}A_j^{-1}$, and do not form a group) are regarded as a mere array connected with this group, and so represented as above by QG_1; and the relation of the original group G to the group Γ_1 (consisting of elements commutative with those of G) and to the new group G_1 is expressed as above by the equation $G = \Gamma_1 . QG_1, = QG_1 . \Gamma_1$.

CAMBRIDGE, *2nd June*, 1893.